On antimagic directed graphs

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Abstract

An antimagic labeling of an undirected graph $G$ with $n$ vertices and $m$ edges is a bijection from the set of edges of $G$ to the integers $\{1, \ldots, m\}$ such that all $n$ vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called antimagic if it admits an antimagic labeling. In [6], Hartsfield and Ringel conjectured that every simple connected graph, other than $K_2$, is antimagic. Despite considerable effort in recent years, this conjecture is still open. In this paper we study a natural variation; namely, we consider antimagic labelings of directed graphs. In particular, we prove that every directed graph whose underlying undirected graph is “dense” is antimagic, and that almost every undirected $d$-regular graph admits an orientation which is antimagic.

1 Introduction

An antimagic labeling of an undirected graph $G$ with $n$ vertices and $m$ edges is a bijection from the set of edges of $G$ to the integers $\{1, \ldots, m\}$ such that all $n$ vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called antimagic if it admits an antimagic labeling. In [6], Hartsfield and Ringel conjectured that every simple connected graph, other than $K_2$, is antimagic. This conjecture is still open; however, some special cases of it were proved. Alon et al. [1] proved that large dense graphs are antimagic; that is, there exists an absolute constant $C > 0$ such that any graph with $n$ vertices and minimum degree at least $C \log n$ is antimagic. Hefetz [7]

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proved that any graph on \( n = 3^k \) vertices that admits a triangle factor is antimagic. This was later generalized by Hefetz et al. [8] to graphs on \( n = p^k \) vertices that admit a \( C_{p^r} \)-factor, where \( p \) is an odd prime. Cranston [3] proved that any regular bipartite graph with minimum degree at least 2 is antimagic. For more related results the reader is referred to the survey [5].

One can define antimagic digraphs in the obvious analogous way. An antimagic labeling of a directed graph \( D \) with \( n \) vertices and \( m \) arcs is a bijection from the set of arcs of \( D \) to the integers \( \{1, \ldots, m\} \) such that all \( n \) oriented vertex sums are pairwise distinct, where an oriented vertex sum is the sum of labels of all arcs entering that vertex minus the sum of labels of all arcs leaving it. A digraph is called antimagic if it admits an antimagic labeling. A natural question to ask would be: Is every simple connected digraph antimagic? Note that the answer is positive for a single directed edge. We would like to rephrase this question in the following equivalent way:

- Is every orientation of any simple connected undirected graph antimagic?

Relaxing this form, we obtain the following question:

- Does every undirected graph admit an antimagic orientation?

In this paper we will provide partial answers to both questions.

Starting with the former question, it is possible to adapt the proof of Alon et al. [1] of the aforementioned result to directed graphs (though this is not entirely straightforward as some technical problems arise). This provides an affirmative answer to the first question for “dense” graphs.

**Theorem 1.1** There exists an absolute constant \( C \) such that the following holds. Let \( G \) be any undirected graph on \( n \) vertices with minimum degree at least \( C \log n \); then every orientation of \( G \) is antimagic.

It is easy to see that, in general, the answer to the first question is “no”. Indeed, both \( K_{1,2} \) and \( K_3 \) admit an orientation which is not antimagic (orient them such that the out-degree and in-degree of every vertex is at most 1), so not every directed graph is antimagic. However, these are the only (connected) counterexamples we have found. It is therefore still possible that every other directed graph whose underlying undirected graph is connected is antimagic. Apart from Theorem 1.1, our results in this setting are fairly modest. In the undirected case, it is straightforward to prove that any graph on \( n \) vertices with a vertex of degree \( n - 1 \) is antimagic (even a vertex of degree \( n - 2 \) suffices [1], though this is no longer trivial). Proving that every orientation of such a graph is antimagic, however, seems rather difficult. We have succeeded for the following special cases:

**Theorem 1.2** For every orientation of every undirected graph that belongs to one of the following families there exists an antimagic labeling.
1. Stars \( S_n \) on \( n+1 \) vertices for every \( n \neq 2 \).

2. Wheels \( W_n \) on \( n+1 \) vertices for every \( n \geq 3 \).

3. Cliques \( K_n \) on \( n \) vertices for every \( n \neq 3 \).

Note that for sufficiently large \( n \), the third part of Theorem 1.2 follows immediately from Theorem 1.1. However, proving the result for every \( n \neq 3 \) requires a different proof.

The second question is easier to tackle. As it is a relaxation of the first question, it can immediately be answered in the positive for the graph classes covered by Theorem 1.1 and 1.2. Moreover, it is easy to see that every bipartite antimagic undirected graph \( G = (A \cup B, E) \) admits an antimagic orientation. Indeed, one can direct all edges from \( A \) to \( B \) and apply any of the antimagic labelings of \( G \) (clearly, this changes only the signs of the vertex sums of the vertices of \( A \)). Using this observation and the aforementioned result of Cranston [3], we immediately conclude that every regular bipartite graph admits an antimagic orientation. We generalize this result as follows:

**Theorem 1.3** Let \( G = (V, E) \) be a \( (2d+1) \)-regular (not necessarily connected) undirected graph with \( d \geq 0 \), then there exists an antimagic orientation of \( G \).

**Theorem 1.4** Let \( G = (V, E) \) be a \( 2d \)-regular undirected connected graph with \( d \geq 1 \). If \( G \) admits a matching that covers all but at most one vertex of \( V \), then there exists an antimagic orientation of \( G \).

**Remark** It seems hard to discard any of the two conditions in Theorem 1.4, that is connectedness and having a matching that covers all vertices but at most one. In fact, we do not even know if every disjoint union of cycles admits an antimagic orientation.

Since for every \( d \geq 3 \) a random \( d \)-regular graph on \( n \) vertices almost surely is connected and admits a matching that covers all but at most one vertex (see e.g. [2]), we immediately obtain the following corollary:

**Corollary 1.5** For every \( d \geq 3 \) and sufficiently large \( n \), almost every \( d \)-regular graph on \( n \) vertices admits an antimagic orientation.

A different approach to generalizing (a directed version of) Cranston’s result [3] is manifested in the following theorem and its immediate corollary.

**Theorem 1.6** Let \( G \) be a graph on \( 2n \) vertices that admits a perfect matching, and let \( U \) be an independent set in \( G \) of size \( n \). If \( \text{deg}(u) \geq 3 \) for every \( u \in U \), then \( G \) admits an antimagic orientation.
Corollary 1.7 Let $G = (A \oplus B, E)$ be a bipartite graph that admits a perfect matching. If $\deg(a) \geq 3$ for every $a \in A$, then $G$ admits an antimagic orientation.

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in theorems we prove. Some of our results are asymptotic in nature and whenever necessary we assume that $n$ is sufficiently large. Throughout the paper, log stands for the natural logarithm. Our graph-theoretic notation is standard and follows that of [4].

The rest of this paper is organized as follows: In Section 2 we prove Theorem 1.1 and 1.2. In Section 3 we prove Theorem 1.3, 1.4 and 1.6. Finally, in Section 4, we present some conclusions and open problems.

2 Graphs for which every orientation is antimagic

In this section we prove Theorem 1.1 and 1.2. Our proof of Theorem 1.1 follows that of Theorem 1.2 from [1], but it also contains some new ideas. We will omit certain details if they are very similar to the ones from [1], and elaborate on the different ones. The main differences between both proofs are in Lemma 2.2 and in Phase 5 of the proof of the theorem.

For a graph $G = (V, E)$ and a vertex $u \in V$, let $\Gamma_G(u) := \{v \in V : (u, v) \in E\}$ denote, as usual, the neighborhood of $u$, and let $\hat{\Gamma}_G(u) := \{(u, v) \in E : v \in \Gamma_G(u)\}$ denote the set of edges which are incident with $u$ in $G$. Often, when there is no risk of confusion, we abbreviate $\Gamma_G(u)$ to $\Gamma(u)$ and $\hat{\Gamma}_G(u)$ to $\hat{\Gamma}(u)$.

Lemma 2.1 There exist absolute positive constants $C_1, C_2$ such that the following holds. Let $t$ and $d$ be integers such that $C_1 \log t \leq d \leq t/30$, and for $1 \leq i \leq d$ let $a_{i,1}, a_{i,2}$ be two distinct elements of $\{1, \ldots, t\}$, chosen randomly with all choices of $d$ pairwise disjoint pairs being equally likely. Let $\sigma \in \{-1, +1\}^d$ be any sign vector. Then, with probability at least $1 - 1/t^2$ the following holds. For every $1 \leq i \leq d$, choose $j_i \in \{1, 2\}$ randomly, independently and uniformly, and consider the random sum $Q = \sum_{i=1}^d \sigma_i a_{i,j_i}$. Then, for every integer $s$, the probability that $Q$ is equal to $s$ is at most $C_2 t \sqrt{d}$.

We omit the proof of Lemma 2.1, as it is a straightforward adaptation of the proof of Lemma 2.3 from [1].

Lemma 2.2 Let $C$ be a positive constant and let $d = \lfloor C \log(n) \rfloor$. For sufficiently large $n$ the following holds. Let $G = (A \oplus B, E)$ be a graph with $n$ vertices and an even number of edges, such that $\deg(v) \in \{d-1, d\}$ for every $v \in A$, $\deg(v) \geq d + 1$ for every $v \in B$, and $B$ is an independent set. Moreover, let $D(G)$ be any orientation of $G$.  

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Then there exists a function $F : B \to 2^E$ that assigns to each $v \in B$ a subset of the edges that are incident with $v$, and a bijective mapping $p : E \to E$ (a pairing of the edges of $G$) such that the following properties hold:

(1) For every $v \in B$ the set $F(v)$ contains an even number of edges; moreover

\[ d + 1 \leq \deg(v) - |F(v)| \leq 2d. \]

(2) For every $v \in B$ we have $p(F(v)) = F(v)$. Moreover, if an edge $e \in \hat{\Gamma}(v)$ is directed away from $v$ in $D(G)$, then $p(e)$ is also directed away from $v$, while if $e$ is directed towards $v$ in $D(G)$, then $p(e)$ is also directed towards $v$.

(3) If $e \in E \setminus F(B)$, where $F(B) := \bigcup_{v \in B} F(v)$, then $e$ and $p(e)$ are independent edges (that is edges that do not share a vertex).

(4) For any two vertices $u, v \in V$ we have $|p(\hat{\Gamma}(u) \setminus F(u)) \cap \hat{\Gamma}(v)| \leq 100$ (if $u \in A$, then we use the convention $F(u) = \emptyset$).

**Proof** We first describe how to define $F$ and $p|_{F(B)}$ successively, such that (1), (2) and (4) are satisfied.

Initially, $F(v) = \emptyset$ for every $v \in B$. In each step, pick a vertex $v \in B$ with $\deg(v) - |F(v)| \geq 2d + 1$ (if no such vertex exists we are done). Partition the edges of $\hat{\Gamma}(v) \setminus F(v)$ into two sets, those that are directed away from $v$ in $D(G)$ and those that are directed towards $v$. Let $T$ be the larger of the two sets (breaking ties arbitrarily). Note that $|T| \geq d + 1$. Choose two edges $(v, x), (v, y)$ from $T$ for which there is no other vertex $w \in B$ such that $(w, x), (w, y) \in F(w)$ and $p((w, x)) = (v, y)$ (this constraint ensures that property (4) holds with a stronger bound of 1 on the right hand side of the inequality). Such a choice is possible, since, if for every two edges $(v, x), (v, y)$ from $T$ there was a vertex $w \in B$ such that $(w, x), (w, y) \in F(w)$ and $p((w, x)) = (w, y)$, then the vertices in $\Gamma(v) \cap V(T) \subseteq A$ would be of a degree which is strictly larger than $d$; clearly, this is a contradiction. Add $(v, x)$ and $(v, y)$ to $F(v)$ and set $p((v, x)) = (v, y)$ and $p((v, y)) = (v, x)$.

Next we will show how to extend $p$ to all elements of $E$, without affecting (1) and (2), such that (3) and (4) are satisfied.

Let $t$ denote the number of edges of $G$. Moreover, let $G' = (V(G), E \setminus F(B))$, and let $k$ denote the number of edges of $G'$. Note that $k = t - |F(B)|$ is even and that $k \geq \max(t - |B|(n - 1), |B|(d + 1)) \geq d^2/3$ for sufficiently large $n$ (for the last estimate we use the bound $t \geq (d - 1)n/2$). Let $L(G')$ denote the line graph of $G'$, and let $\overline{L}(G')$ denote its complement. Since $\Delta(G') \leq 2d$, it follows that $\delta(\overline{L}(G')) \geq k - 1 - 2(2d - 1) \geq \frac{k}{2} = \frac{1}{2} |V(\overline{L}(G'))|$ holds for sufficiently large $n$. We can thus apply Dirac's theorem to conclude that there exists a Hamilton cycle in $\overline{L}(G')$, and in particular, also a perfect matching (recall that $k$ is even). This immediately defines an extension of the mapping $p$ to the edges of $G'$ such that, for every edge $e \in E(G')$, $e$ and $p(e)$ are independent.
Phase 2: Apply Lemma 2.2 to the graph $G$. Hence, in order to complete the proof, it suffices to show how one can modify $p|_{E(G')}$, such that the number of pairs of vertices $u, v \in V$ for which $|\{(e^*, p(e^*)) : e^* \in \hat{\Gamma}_G(u) \cap p(e^*) \in \hat{\Gamma}_G(v)\}| \geq 100$ can be reduced by one (if for all pairs of vertices the cardinality of this set is bounded by 99, then, together with the contribution of $+1$ we get from the definition of $p|_{F(B)}$ at the beginning of the proof, we can ensure that the inequality in condition (4) holds). Let $u$ be a vertex and let $e \in \hat{\Gamma}_G(u)$ be an edge, such that $e' := p(e)$ shares an endpoint $v$ with at least 99 more edges from $p(\hat{\Gamma}_G(u))$. Let $P_{uv}$ denote the set of edge pairs $\{(e^*, p(e^*)) : e^* \in E(G')\}$ that share an endpoint with $e$ or with $e'$; clearly, $|P_{uv}| \leq 8d$. For every vertex $z \in V(G')$, let $S_z$ denote the set of all vertices that have a degree of at least 99 in the graph spanned by the edges of $p(\hat{\Gamma}_G(z))$. Note that for every $z \in V(G')$, we have $|p(\hat{\Gamma}_G(z))| \leq 2d$ and thus $|S_z| \leq 4d/99$. Let $Q_{uv}$ be the set of edge pairs $\{(e^*, p(e^*)) : e^* \in E(G')\}$ for which either $e^*$ or $p(e^*)$ has an endpoint in $S_u$ or in $S_v$. Note that $|Q_{uv}| \leq 2d(|S_u| + |S_v|) \leq 16d^2/99$. Pick an arbitrary edge pair $\{e_1, e_2\}$ from $\{(e^*, p(e^*)) : e^* \in E(G')\} \setminus (P_{uv} \cup Q_{uv})$ (the cardinality of this set is at least $k/2 - (8d + 16d^2/99) \geq d^2/6 - 16d^2/99 - 8d \geq 1$ for sufficiently large $n$) and change the pairing of the edges according to $p(e) := e_1$ and $p(e') := e_2$. Repeat this process until $|\{(e^*, p(e^*)) : e^* \in \hat{\Gamma}_G(u) \cap p(e^*) \in \hat{\Gamma}_G(v)\}|$ drops below 100. □

Proof of Theorem 1.1 Let $C$ be a sufficiently large absolute constant, such that for every sufficiently large positive integer $n$ and for every $t \in [dn/2 - 1, dn]$, where $d = |C \log n|$, we have $|C_1 \log t| \leq d - 1$ and $2d \leq t/30$, where $C_1$ is the constant from Lemma 2.1. In order to prove Theorem 1.1 it suffices to prove that for every sufficiently large $n$, if $G = (V, E)$ is a graph with $n$ vertices and $m$ edges, $\delta(G) \geq d$, and $D(G)$ is any orientation of $G$, then $D(G)$ admits an antimagic labeling.

It is convenient to split the description of the proof into five phases.

Phase 1: As long as there are two adjacent vertices in $G$, each having degree at least $d + 1$, assign the largest yet unused label to the edge that connects them and delete it. If the remaining graph contains an odd number of edges, pick some arbitrary edge, label it with the largest yet unused label and delete it. Let $G'$ denote the spanning subgraph of $G$ obtained at the end of this process. Each vertex $v \in V$ has a partial vertex sum, denoted by $r(v)$, which is the oriented sum of the labels of all edges from $\hat{\Gamma}(v)$ that were deleted during this phase (if no edge incident with $v$ was labeled, then we set $r(v) = 0$). Let $A$ denote the set of vertices of $G'$ with degree $d - 1$ or $d$, and let $B = V \setminus A$. Note that the vertices of $B$ form an independent set, and that $\deg_{G'}(v) \geq d + 1$ for every $v \in B$. Let $t \leq m$ denote the number of edges of $G'$. Note that $t$ is even and that $t \in [dn/2 - 1, dn]$. Our goal is to assign the labels $\{1, \ldots, t\}$ to the edges of $G'$ such that all vertex sums (cumulated with the partial sums $\{r(v) : v \in V\}$) will be distinct.

Phase 2: Apply Lemma 2.2 to the graph $G'$ to obtain a function $F$ and a pairing $p$ that satisfy conditions (1)–(4) of the lemma.

Phase 3: Randomly partition the set $\{1, \ldots, t\}$ into $t/2$ pairwise disjoint pairs of labels.
Assign those pairs of labels arbitrarily to the edge pairs determined by $p$. For every $e \in E(G')$ let $L(e)$ denote the pair of labels assigned to the edge pair $\{e, p(e)\}$.

**Phase 4:** For each $v \in B$ let $f(v)$ denote the oriented vertex sum of the labels assigned to the edges of $F(v)$. Note that although we have not yet specified which edge will get which label (for each edge there are two choices), $f(v)$ is well-defined by condition (2) from Lemma 2.2. For every $v \in A$ we set $f(v) = 0$. For every $v \in B$ let $H(v) = \Gamma_G(v) \setminus F(v)$, and for every $v \in A$ let $H(v) = \Gamma_G(v)$. Note that $d - 1 \leq |H(v)| \leq 2d$ (condition (1) from Lemma 2.2). For every vertex $v \in V$ there are $2^{|H(v)|}$ different ways of assigning the labels from $L(H(v))$ to the edges of $H(v)$, each assignment yielding a certain contribution $Q(v)$ to the oriented vertex sum of $v$ (note condition (3) from Lemma 2.2 here). By Lemma 2.1, for each fixed vertex $v$ with probability at least $1 - 1/t^2$, $Q(v)$ attains no integer $s$ for more than a $C_2/(t|H(v)|^{1/2})$-fraction of the $2^{|H(v)|}$ assignments. It follows from a union bound argument that one can fix an assignment of label pairs to edge pairs, such that for every vertex $v$ and for every integer $s$, we have $\Pr[Q(v) = s] \leq \frac{C_2}{t|H(v)|^{1/2}} \leq \frac{C_2}{t \sqrt{d - 1}}$.

**Phase 5:** For every pair $\{e, p(e)\} \subseteq E(G')$ we flip a fair coin to decide which edge gets which label from $L(e)$; all $t/2$ coin flips are independent. Note that the final weight of each vertex $v$ is given by $r(v) + f(v) + Q(v)$. We claim that with positive probability no two vertices will end up with the same final weight. For every vertex $v$ let $J(v) \in \{1, 2\}^{|H(v)|}$ denote the decision vector, showing the way in which the labels from $L(H(v))$ are assigned to every edge $e \in H(v)$ and its paired edge $p(e)$. For any two vertices $u, v \in V$ such that exactly $l$ edges from $H(u) \cup p(H(u))$ have $v$ as an endpoint and for every decision vector $\bar{x} \in \{1, 2\}^{|H(u)|}$ we have $\Pr[Q(v) = s | J(u) = \bar{x}] \leq 2^{l} \Pr[Q(v) = s]$, for every integer $s$. By our choice of the pairing function $p$ (see condition (4) from Lemma 2.2) we have $l \leq 101$ (if $u$ and $v$ are not adjacent, then $l \leq 100$), and thus the right hand side of this inequality is bounded from above by $2^{101} \frac{C_2}{t \sqrt{d - 1}} \leq 2^{101} \frac{C_2}{(dn/2 - 1) \sqrt{d - 1}}$. For every two vertices $u, v \in V$ let $B(u, v)$ denote the event that both $u$ and $v$ end up with the same final weight. Denoting $s(u, v) := r(u) + f(u) - r(v) - f(v)$ we have

$$\Pr[B(u, v)] = \Pr[Q(v) = Q(u) + s(u, v)] = \sum_{\bar{x} \in \{1, 2\}^{|H(u)|}} \Pr[J(u) = \bar{x}] \cdot \Pr[Q(v) = Q(u) + s(u, v) | J(u) = \bar{x}] \leq 2^{101} \frac{C_2}{(dn/2 - 1) \sqrt{d - 1}}.$$

We will prove that $B(u, v)$ is independent of all other events $B(x, y)$ ($x, y \in V$) but at most $O(nd)$. Let $Z$ denote the set of vertices that are endpoints of edges of $H(u) \cup H(v) \cup p(H(u)) \cup p(H(v))$. Clearly, $|Z| \leq 6 \cdot 2d + 2$. Any event $B(x, y)$ where neither $x$ nor $y$ is in $Z$ is independent of $B(u, v)$. Thus $B(u, v)$ is independent of all but at most $(12d + 2)n$ other events. Since

$$2^{101} \frac{C_2}{(dn/2 - 1) \sqrt{d - 1}} \cdot (12d + 2)n \leq \frac{1}{3},$$

we have

$$\Pr[B(u, v) | Z^c] \leq \frac{1}{3}.$$
holds for sufficiently large \( n \), we can apply the Lovász Local Lemma and conclude that with positive probability no two oriented vertex sums are the same. It follows that \( D(G) \) admits an antimagic labeling.

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\square
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Our next goal is to prove Theorem 1.2. Before doing so, we state and prove a lemma that will be used when proving the third part of the theorem. It might also be of independent interest. This lemma determines precisely the values \( n, m \in \mathbb{N} \) for which there exists a simple graph \( G \) with \( n \) vertices and \( m \) edges, such that the degree of every vertex of \( G \) is even (we will call such graphs even degree graphs). It turns out that such graphs exist for almost all pairs \((n, m)\) with \( 0 \leq m \leq \binom{n}{2} \).

**Lemma 2.3** Let \( n \in \mathbb{N} \) and let \( m \in \{0, 1, \ldots, \binom{n}{2}\} \). There exists a simple even degree graph with \( n \) vertices and \( m \) edges, if and only if \( m \notin \{1, 2, \binom{n}{2} - \frac{n}{2} + 1, \binom{n}{2} - \frac{n}{2} + 2, \ldots, \binom{n}{2}\} \) for odd \( n \), and \( m \notin \{1, 2, \binom{n}{2} - \frac{n}{2} + 1, \binom{n}{2} - \frac{n}{2} + 2, \ldots, \binom{n}{2}\} \) for even \( n \).

**Proof** Clearly, no single edge and no pair of edges can comprise an even degree graph. On the other hand, if \( n \) is odd, then \( G \subseteq K_n \) is an even degree graph if and only if \( K_n \setminus G \) is an even degree graph; this excludes the values \( m \in \{\binom{n}{2} - 2, \binom{n}{2} - 1\} \). If \( n \) is even, then all vertices of \( K_n \) have odd degree and at least \( n/2 \) edges (a perfect matching) have to be removed from \( K_n \) to yield an even degree graph; this excludes the values \( m \in \{\binom{n}{2} - \frac{n}{2} + 1, \binom{n}{2} - \frac{n}{2} + 2, \ldots, \binom{n}{2}\} \).

In order to prove that there exists an even degree graph for all of the remaining values of \( m \), we proceed by induction on \( n \). The claim is easily verified for all \( n \leq 5 \).

Let \( n > 5 \) and assume first that \( n \) is even. By the induction hypothesis there exists an even degree graph on \( n - 1 \) vertices with exactly \( m \) edges for every \( m \in \{0, 3, 4, \ldots, \binom{n-1}{2} - 4, \binom{n-1}{2} - 3, \binom{n-1}{2} - 2\} \). Adding an isolated vertex to one of these graphs yields an even degree graph on \( n \) vertices; this proves our claim for these values of \( m \). Let \( G \) be an even degree graph on \( n - 1 \) vertices with exactly \( \binom{n-1}{2} - 3 \) edges. Remove an arbitrary edge \((x, y)\) from \( G \), and add a new vertex \( u \) and the two edges \((u, x), (u, y)\) to obtain an even degree graph on \( n \) vertices with exactly \( \binom{n-1}{2} - 2 \) edges. Starting with \( K_n \), remove a perfect matching \((u_1, v_1), \ldots, (u_{n/2}, v_{n/2})\) to obtain an even degree graph \( H \) on \( n \) vertices with exactly \( \binom{n}{2} - n/2 \) edges. Let \( 1 \leq k \leq n/2 \). For every \( 1 \leq i \leq k \), replace the pair of edges \((u_i, u_{i+1}), (v_i, u_{i+1})\) of \( H \) with the single edge \((u_i, v_i)\) to obtain an even degree subgraph of \( K_n \) with exactly \( \binom{n}{2} - n/2 - k \) edges. The case of even \( n \) now follows, as \( \{0, 3, 4, \ldots, \binom{n-1}{2} - 4, \binom{n-1}{2} - 3, \binom{n-1}{2} - 2\} \cup \{\binom{n}{2} - n, \ldots, \binom{n}{2} - n/2\} = \{0, 3, 4, \ldots, \binom{n}{2} - n/2\} \).

Assume now that \( n > 5 \) is odd. As in the previous case, an application of the induction hypothesis and the addition of an isolated vertex yields even degree graphs \( G_m \) on \( n \) vertices with exactly \( m \in \{0, 3, 4, \ldots, \binom{n-1}{2} - \frac{n-1}{2}\} \) edges. Since \( n \) is odd, \( K_n \setminus G_m \) is also an
even degree graph; clearly, it has exactly $\binom{n}{2} - m$ edges. The case of odd $n$ now follows as well since $\{0, 3, 4, \ldots, \binom{n-1}{2} - \frac{n-1}{2}\} \cup \{\binom{n}{2} - \binom{n-1}{2} + \frac{n-1}{2}, \ldots, \binom{n}{2} - 4, \binom{n}{2} - 3, \binom{n}{2}\} = \{0, 3, 4, \ldots, \binom{2}{2} - 4, \binom{2}{2} - 3, \binom{2}{2}\}$ holds for $n > 5$. \hfill \Box

Proof of Theorem 1.2

1. The claim holds trivially for $n = 1$, so we can assume that $n \geq 3$. Let $D(S_n)$ be an arbitrary orientation of $S_n$. Let $u$ denote the center of the star (the unique vertex of degree $n$). Let $I = \{x_1, \ldots, x_r\}$ denote the set of in-neighbors of $u$ (that is, $(x_i, u) \in D(S_n)$ for every $1 \leq i \leq r$) and let $O = \{x_{r+1}, \ldots, x_n\}$ denote the set of out-neighbors of $u$. Assume without loss of generality that $r \geq n/2$ (reversing all edges of a graph results in multiplying every vertex-sum by $-1$; this preserves antimagic labelings). For every $1 \leq i \leq n$ assign the edge connecting $u$ and $x_i$ the label $n + 1 - i$. Denote the resulting vertex sum of a vertex $v$ by $\omega(v)$, then clearly

$$\omega(x_1) < \cdots < \omega(x_r) < \omega(x_n) < \cdots < \omega(x_{r+1}) < \omega(u),$$

where the last inequality follows since $r \geq n/2$ and $n \geq 3$.

2. For $3 \leq n \leq 10$ we have verified the claim by exhaustive enumeration, using a computer; hence we will assume that $n \geq 11$. Let $D(W_n)$ be an arbitrary orientation of $W_n$. Let $u$ denote the center of the wheel. Let $I = \{x_1, \ldots, x_r\}$ denote the set of in-neighbors of $u$ and let $O = \{x_{r+1}, \ldots, x_n\}$ denote the set of out-neighbors of $u$. Assume without loss of generality that $r \geq n/2$. We will also assume for now that $r \leq n - 3$ and treat the other cases separately afterwards. Assume for convenience that $n$ is even (for odd $n$ the proof is essentially the same). Assign the “cycle edges” (edges which are not incident with $u$) of $D(W_n)$ the labels $1, \ldots, n$ in the following clockwise order:

$$1, n, 2, n-1, 3, n-2, \ldots, n/2, n/2 + 1.$$

Denote the resulting vertex sum of a vertex $v$ by $\omega(v)$; clearly, $-(n+2) \leq \omega(x_i) \leq n+2$ for every $1 \leq i \leq n$. Assume without loss of generality that $\omega(x_1) \leq \cdots \leq \omega(x_r)$ and that $\omega(x_{r+1}) \geq \cdots \geq \omega(x_n)$. Assign the edge connecting $u$ and $x_i$ the label $2n + 1 - i$ and denote the resulting vertex sum of a vertex $v$ by $\hat{\omega}(v)$. It is easy to see that the following conditions hold:

(a) $\hat{\omega}(x_1) < \cdots < \hat{\omega}(x_r)$ and $\hat{\omega}(x_{r+1}) > \cdots > \hat{\omega}(x_n)$;

(b) $\hat{\omega}(x_i) \leq -2$ for every $1 \leq i \leq r$ (since $r \leq n - 3$);

(c) $\hat{\omega}(x_i) \geq -1$ for every $r + 1 \leq i \leq n$;

(d) $\hat{\omega}(x_i) \leq 2.5n + 2$ for every $1 \leq i \leq n$ (since $r \geq n/2$).

In order to prove that all vertex sums are distinct, it is therefore sufficient to prove that $\hat{\omega}(u) \neq \hat{\omega}(x_i)$ for every $1 \leq i \leq n$; by (d) above it suffices to prove that
\[\hat{\omega}(u) > 2.5n + 2.\] We have
\[
\hat{\omega}(u) = \sum_{i=1}^{r} (2n+1-i) - \sum_{j=r+1}^{n} (2n+1-j) = (2n+1)(2r-n) + \left(\frac{n+1}{2}\right) - 2\left(\frac{r+1}{2}\right).
\]

It is clear that \(\hat{\omega}(u)\) is minimized for \(r = n/2\), entailing \(\hat{\omega}(u) \geq n^2/4 > 2.5n + 2\), where the last inequality holds for \(n \geq 11\).

The cases \(r = n\) and \(r = n - 1\) are easy. The case \(r = n - 2\) is slightly more involved, but still easy. The main idea is to label the cycle edges that are incident with \(x_{n-1}\) and with \(x_n\) with the labels \(1, 3, 2\) if they are adjacent, and with the labels \(1, 4\) and \(2, 3\) respectively, otherwise. The rest of the cycle edges will be labeled similarly to the proof above, entailing \(-5 \leq \omega(x_{n-1}), \omega(x_n) \leq 5\), and \(-(n+6) \leq \omega(x_i) \leq n + 6\) for every \(1 \leq i \leq n - 2\). This is enough to ensure \(\hat{\omega}(x_1) < \cdots < \hat{\omega}(x_{n-2}) < \hat{\omega}(x_n) < \hat{\omega}(x_{n-1}) < \hat{\omega}(u)\).

3. For \(n \in \{1, 2, 4, 5\}\) the claim can be easily verified; thus we can assume that \(n \geq 6\). Let \(D(K_n)\) be an arbitrary orientation of \(K_n\). Let \(u\) be a vertex of \(D(K_n)\) with maximum in-degree, let \(I = \{x_1, \ldots, x_r\}\) denote the set of in-neighbors of \(u\) and let \(O = \{x_{r+1}, \ldots, x_{n-1}\}\) denote the set of out-neighbors of \(u\). Note that \(r \geq \left\lceil \frac{n-1}{2} \right\rceil\) by our assumption of maximality. Let \(L_I\) denote the set of the largest \(r\) labels of the same parity as \(\binom{n}{2}\), and let \(L_O\) denote the set of the smallest \(n-1-r\) labels of the opposite parity. Assign the labels of \(\{1, 2, \ldots, \binom{n}{2}\}\) \(L_I\cup L_O\) to the edges of \(D(K_n)\setminus\{u\}\), such that the edges which are assigned odd labels span an even degree subgraph (such an even degree subgraph exists due to Lemma 2.3, which applies since \(n \geq 6\) and thus the number of odd labels in \(\{1, 2, \ldots, \binom{n}{2}\}\setminus\{L_I\cup L_O\}\) is strictly larger than 2, and clearly also small enough). Denote the resulting vertex sum of a vertex \(v\) by \(\omega(v)\). Assume without loss of generality that \(\omega(x_1) \leq \cdots \leq \omega(x_r)\) and that \(\omega(x_{r+1}) \leq \cdots \leq \omega(x_{n-1})\). Note that \(\omega(v)\) is even for every vertex \(v \in V(K_n)\). Assign the labels from \(L_I\) in descending order to the edges \((x_1, u), \ldots, (x_r, u)\) and the labels from \(L_O\) in ascending order to the edges \((u, x_{r+1}), \ldots, (u, x_{n-1})\). Denote the resulting vertex sum of a vertex \(v\) by \(\hat{\omega}(v)\). The following conditions hold:

\(\hat{\omega}(x_1) < \cdots < \hat{\omega}(x_r)\) and \(\hat{\omega}(x_{r+1}) < \cdots < \hat{\omega}(x_{n-1})\);

\((b)\) For every \(1 \leq i \leq r\) and every \(r+1 \leq j \leq n-1\) we have \(\hat{\omega}(x_i) \neq \hat{\omega}(x_j) \text{ (mod 2)}\);

in particular \(\hat{\omega}(x_i) \neq \hat{\omega}(x_j)\);

In order to prove that all vertex sums are distinct, it is therefore sufficient to prove that \(\hat{\omega}(u) > \hat{\omega}(x_i)\) for every \(1 \leq i \leq n - 1\). Recall that \(u\) is a vertex with maximum in-degree \(r\); thus, by our labeling \(\hat{\omega}(u) \geq \hat{\omega}(x_i)\) for every \(1 \leq i \leq n - 1\). Assume for the sake of contradiction that there exists some \(1 \leq i \leq n - 1\) for which \(\hat{\omega}(u) = \hat{\omega}(x_i)\).

It follows that \(u\) and \(x_i\) have the same in-degree \(r = \frac{n-1}{2}\), that
\[
\hat{\omega}(u) = \sum_{i=0}^{r-1} \left(\binom{n}{2} - 2i\right) - \sum_{i=1}^{r} 2i,
\]
and that
\[ \hat{\omega}(x_i) = \sum_{i=0}^{r-1} \left( \binom{n}{2} - 2i - 1 \right) - \sum_{i=1}^{r} (2i - 1). \]

This is clearly impossible, as the label of the edge \((u, x_i)\) has to appear in both sums. 

\[\square\]

3 Graphs that admit an antimagic orientation

The main idea of the proofs of Theorem 1.3 and 1.4 is to use eulerian orientations. To start with the simplest case, given a eulerian graph, one can orient all edges along a eulerian cycle, and label them in consecutive increasing order along this cycle. If the graph is regular this gives a labeling where all vertex sums except one (the start and end vertex of the cycle) are the same. We then perturb this “almost magic” labeling and obtain an antimagic labeling by flipping the orientation of certain arcs or removing them. Depending on which type of graph we start with, different methods on how to make it eulerian and how to perturb an “almost magic” labeling are used.

Proof of Theorem 1.3

Let \(2n\) denote the number of vertices and \(m\) the number of edges of \(G\). Let \(M = \{(u_i, v_i) : 1 \leq i \leq n\}\) be a perfect matching of the vertices of \(V\) such that \(H := G \oplus M\) is a connected multigraph (it might contain parallel edges); such a matching \(M\) exists as every connected component of \(G\) is of size at least 2. Let \(D(H)\) be an arbitrary eulerian orientation of \(H\); assume without loss of generality that \((u_i, v_i)\) is directed from \(u_i\) to \(v_i\) in \(D(H)\) for every \(1 \leq i \leq n\). We consider the eulerian cycle of \(D(H)\) as if it ends with the edge \((u_1, v_1)\); clearly, this means that the cycle starts with some edge \((v_1, w)\) which is directed from \(v_1\) to \(w\) in \(D(H)\). Label the edges of \(H\) according to the ordering dictated by this eulerian orientation as follows. The edge \((v_1, w)\) will get the label 1. For every other edge, if it is an edge of \(M\), it will get the same label as the previous edge, whereas if it is an edge of \(G\), it will get the successor of that label. Denote this edge labeling by \(f\). It is evident that \(f\), when restricted to the edges of \(G\), is a bijection from \(E\) to the integers \(\{1, \ldots, m\}\) and that the restriction of \(f\) to the edges of \(M\) is injective. Moreover, the current vertex sum of \(v_1\) is \(m - (d + 1)\), the current vertex sum of \(v_i\) is \(-(d + 1)\) for every \(2 \leq i \leq n\), and the current vertex sum of \(u_i\) is \(-d\) for every \(1 \leq i \leq n\). Remove the edges of \(M\) and denote the resulting vertex sum of \(x \in V\) by \(\omega(x)\). For \(1 \leq i \leq n\) let \(a_i\) denote the label that was assigned to the edge \((u_i, v_i)\). It is easy to see that \(\omega(v_1) = -(d + 1)\), \(\omega(v_i) = -d + a_i < \omega(v_1)\) for every \(2 \leq i \leq n\) and \(\omega(u_i) = -d + a_i > \omega(v_1)\) for every \(1 \leq i \leq n\). It follows that the restriction of \(f\) to the edges of \(G\) is an antimagic labeling of \(D(G)\).
Proof of Theorem 1.4

Let $M = \{(u_i, v_i) : 1 \leq i \leq n\}$ be a matching that covers all vertices of $G$, but at most one. Note that the number of vertices of $G$ is either $2n$ or $2n+1$.

First consider the case that $M$ is perfect. Let $D(G)$ be an arbitrary eulerian orientation of $G$. Starting with the edge $(u_1, v_1)$, label the edges of $G$ with the numbers $1, \ldots, 2dn$ in increasing order along a eulerian cycle of $D(G)$. The current vertex sum of $u_1$ is $d(2n-1)$, whereas the current vertex sum of any other vertex is $-d$. Now switch the orientation of all edges of $M$. Observe that switching the orientation of an edge with the label $a$ changes the vertex sums of its endpoints by $+2a$ and $-2a$, respectively. For every vertex $v \in V$ let $\omega(v)$ denote the new vertex sum of $v$. As the labels of the matching edges differ by at least 2, for every $u, v \in V \setminus \{u_1\}$ we have $|\omega(u) - \omega(v)| \geq 4$; in particular, all of these vertex sums are distinct. Hence there is only one potential conflict, a conflict between the vertex sums at $u_1$ and at some $v \in V \setminus \{u_1\}$. In this case, switch back the orientation of the edge $(u_1, v_1)$. This changes the vertex sums of $u_1$ and $v_1$ by $-2$ and $+2$ back to $d(2n-1)$ and $-d$ and thus resolves all conflicts.

Now suppose one vertex $s$ remains unmatched by $M$. Let $D(G)$ be a eulerian orientation of $G$ that admits a eulerian cycle $C$ in which the edges $(s, u_1)$ and $(u_1, v_1)$ appear consecutively (it is not hard to see that such a eulerian orientation exists; possibly one will have to rename matching edges). Starting with the edges $(s, u_1)$ and $(u_1, v_1)$, label the edges of $G$ with the numbers $1, \ldots, d(2n+1)$ in increasing order along the eulerian cycle $C$ of $D(G)$. The current vertex sum of $s$ is $2dn$, whereas the current vertex sum of any other vertex is $-d$. Now switch the orientation of all edges of $M$. For every vertex $v \in V$ let $\omega(v)$ denote the new vertex sum of $v$. Note that the edge $(u_1, v_1)$ is assigned the label 2, and thus we have $\omega(u_1) = -d + 4$. For every $u, v \in V \setminus \{s\}$ we have $|\omega(u) - \omega(v)| \geq 4$; in particular, all of these vertex sums are distinct. Hence there is only one potential conflict, a conflict between the vertex sums at $s$ and at some $v \in V \setminus \{s\}$. In this case switch the orientation of the edge $(s, u_1)$. This changes the vertex sums of $s$ and $u_1$ by $+2$ and $-2$ to $2dn + 2$ and $-d + 2$ and thus resolves all conflicts.

\[ \square \]

In the proof of Theorem 1.6 we will use the following two technical lemmas. The first one is due to Kaplan, Lev and Roditty, and is cited here without proof.

**Lemma 3.1** ([9]) Let $r$ be a positive integer and let $r = d_1 + \cdots + d_n$ be a partition of $r$ with $d_i \geq 2$ for every $1 \leq i \leq n$. Define $r' := r + 1$ if $r$ is even and $r' := r$ if $r$ is odd. Then the set $\{1, \ldots, r\}$ can be partitioned into pairwise disjoint sets $X_1, \ldots, X_n$, such that for every $1 \leq i \leq n$ we have $|X_i| = d_i$ and $\sum_{x \in X_i} x \equiv 0 \pmod{r'}$. 

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Lemma 3.2 Let $a \in \mathbb{N}$. Any graph $G = (V, E)$ with $m$ edges admits an orientation and an edge-labeling with the labels $\{a + 1, a + 2, \ldots, a + m\}$, such that every oriented vertex sum is at most $a + m$.

Proof If $m = 0$, then there is nothing to prove; hence we may assume that $m \geq 1$. First, add a set $E'$ of auxiliary edges to $G$ to obtain a (multi)graph $G' = (V, E \cup E')$ such that $G'$ is eulerian (that is, $G'$ is connected and every vertex of $G'$ has an even degree), every vertex $v \in V$ is incident with at most two edges of $E'$, and there exists a vertex $s \in V$ that is incident with at most one edge of $E'$. It is easy to see that one may choose an eulerian orientation $D$ of $G'$ such that, if two edges from $E'$ are incident with a vertex $v$, then exactly one of them is directed towards $v$ and if $s$ is incident with one of the edges of $E'$, then this edge is directed towards $s$. This implies that in the subgraph $H$ of $D$ that is spanned by the edges of $E'$ every vertex $v \in V$ has out-degree at most one.

Now we label the edges of $G$ along the eulerian cycle of $D(G')$, using the labels $\{a + 1, a + 2, \ldots, a + m\}$ in increasing order for the edges of $G$, and reusing the label of the previous edge whenever we encounter an edge of $E'$. We start this labeling at the vertex $s$, such that, if there is an edge of $E'$ incident with $s$, then it is the last edge of the cycle (if $\deg_H(s) = 0$, then just start with any edge incident with $s$).

Denote the resulting vertex sum of every vertex $v$ of $G'$ by $\omega'(v)$. Denote by $\omega(v)$ the partial oriented vertex sum of $v$, obtained by considering only the labels given to edges of $G$.

As the labels along the eulerian cycle of $D(G')$ are monotone increasing, we have $\omega'(v) \leq 0$ for every $v \in V \setminus \{s\}$ and $\omega'(s) = -(\deg_{G'}(s)/2 - 1) - (a + 1) + (a + m) \leq a + m$. As the out-degree in $H$ of every vertex $v \in V$ is at most one, we have $\omega(v) \leq \omega'(v) + (a + m) \leq a + m$ for every $v \in V \setminus \{s\}$. Moreover, $\omega(s) \leq \omega'(s) \leq a + m$. □

Remark The bound $a + m$ in Lemma 3.2 is tight. Consider for instance a graph consisting only of a single edge (or more generally, a matching), then there is essentially only one possible orientation and labeling, and one of the vertices will receive the vertex sum $a + m$.

Proof of Theorem 1.6

Let $m$ denote the number of edges of $G$, let $M$ be a perfect matching of $G$, let $u_1, \ldots, u_n$ be the vertices of the independent set $U$, and let $W = V \setminus U$ (note that $\Gamma(U) = W$). Set $d_i := \deg(u_i) - 1$ for every $1 \leq i \leq n$ and $r := d_1 + \cdots + d_n$; note that $d_i \geq 2$ for every $1 \leq i \leq n$. We will label the non-matching edges that are incident with $U$ using the smallest possible labels $\{1, \ldots, r\}$. The largest $n$ labels $\{m - n + 1, \ldots, m\}$ will be reserved for the edges of $M$ and the remaining labels $\{r + 1, \ldots, m - n\}$ for the edges of $G' := G[W]$.

By Lemma 3.2 one can orient and label the edges of $G'$, using the labels $\{r + 1, \ldots, m - n\}$, such that the vertex sum at every vertex $v \in V(G')$ is at most $m - n$.

Next, orient all edges that are incident with $U$ (including the edges of $M$) towards $U$. By
Lemma 3.1 there is a partition of the set \{1, \ldots , r\} into pairwise disjoint sets \(X_1, \ldots , X_n\) such that for all \(1 \leq i \leq n\) we have \(|X_i| = d_i\) and \(\sum_{x \in X_i} x \equiv 0 \pmod{r'}\) where \(r'\) is defined as in the lemma. Use the labels of \(X_i\) for the non-matching edges incident with \(u_i\) arbitrarily.

Denote the resulting vertex sum of a vertex \(v\) by \(\omega(v)\). The function \(\omega\) defines an ordering \(v_1, \ldots , v_n\) of the vertices of \(W\) such that, without loss of generality, \(\omega(v_1) \geq \cdots \geq \omega(v_n)\).
For every \(1 \leq i \leq n\) assign the label \(m - n + i\) to the matching edge which is incident with \(v_i\). Denote the resulting vertex sum of a vertex \(v\) by \(\hat{\omega}(v)\). Note that the following conditions hold:

1. \(\hat{\omega}(v_i) \leq \omega_{G'}(v_i) - (m - n + 1) < 0\) (\(\omega_{G'}(v_i)\) denotes the partial oriented vertex sum of \(v_i\), obtained by considering only the labels given to edges of \(G'\)), and \(\hat{\omega}(u_i) > 0\) for every \(1 \leq i \leq n\);
2. \(\hat{\omega}(v_1) > \cdots > \hat{\omega}(v_n)\);
3. As \(n < r \leq r'\) we have \(m - n + i \not\equiv m - n + j \pmod{r'}\) for all \(1 \leq i < j \leq n\). Since \(\omega(u_i) \equiv \omega(u_j) \equiv 0 \pmod{r'}\), it follows that \(\hat{\omega}(u_i) \not\equiv \hat{\omega}(u_j) \pmod{r'}\), and therefore \(\hat{\omega}(u_i) \neq \hat{\omega}(u_j)\) for all \(1 \leq i < j \leq n\).

It follows that all vertex sums are distinct.

\[ \square \]

4 Concluding remarks and open problems

All orientations antimagic: The only connected directed graphs that we know are not antimagic are \(K_{1,2}\) and \(K_3\), directed such that the out-degree and the in-degree of every vertex are at most 1. This motivates us to ask the following question:

**Question 4.1** Is every connected directed graph with at least 4 vertices antimagic?

By exhaustive enumeration using a computer we have verified that every connected directed graph on \(4 \leq n \leq 7\) vertices admits an antimagic labeling. This provides some support for an affirmative answer to Question 4.1.
Existence of antimagic orientations: It follows immediately from Theorem 1.1 that every “dense” graph admits an antimagic orientation (in fact, many such orientations). Moreover, we have proved that many “sparse” graphs (including almost all regular graphs) admit an antimagic orientation. This leads us to make the following conjecture:

Conjecture 4.2 Every connected undirected graph admits an antimagic orientation.

Directed and undirected antimagicness: As was indicated in the Introduction, if a bipartite graph admits an antimagic labeling, then it also admits an antimagic orientation (that is, in order to prove that a particular bipartite digraph is antimagic, it suffices to prove that its underlying undirected graph is antimagic). If the assertion of Conjecture 4.2 is true, then this implication holds trivially for every connected graph. However, this does not seem to be an easy claim to prove. Indeed, a natural way to prove such an implication would be to fix an antimagic labeling of the undirected graph, and then to somehow find an orientation which is antimagic with the same labeling. While this can be done with bipartite graphs, it is not possible in general. Consider for example an undirected graph $G$, consisting of $r \geq 5$ vertex disjoint triangles. For every $1 \leq i \leq r$, assign the labels $3i, 3i - 1, 3i - 2$ to the edges of the $i$th triangle arbitrarily. It is easy to see that this yields an antimagic labeling of $G$. Now, let $D$ be any orientation of $G$. In every triangle of $D$ there is a vertex of out-degree exactly one. Its oriented vertex sum must lie in $\{-2, -1, 1, 2\}$. Since $r \geq 5$, there will be two vertices of $D$ with the same vertex sum. This example shows that there exists an antimagic labeling $f$ of an undirected graph $G$ such that $f$ is not an antimagic labeling of any orientation $D$ of $G$.

References


