A New Proof of the Colored Branch Theorem

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Abstract—The colored branch theorem is a result from graph theory that has been described first by Minty. It states the existence of certain meshes and cuts in a graph, whose edges are colored red, green and blue, respectively. The theorem, sometimes also referred to as the lemma of the colored arcs, can be utilized to give short and elegant proofs of many other theorems in graph and circuit theory and has therefore turned out to be of vital importance. We present a new set theoretic proof of the colored branch theorem, that reveals its relationship to the orthogonality theorem, another well-known fundamental result about meshes and cuts in a graph.

I. INTRODUCTION

Of special interest for the electrical engineer is the colored branch theorem, a result that has been described first by Minty in 1960 [1]. Its importance for circuit theory arises from its proposition about the existence of certain meshes and cuts in an edge-colored directed graph, since meshes and cuts are basic terms of a mathematical analysis of electrical networks. Besides a purely graph theoretic formulation the theorem can be illustrated by some interesting and convincing analogies in terms of electrical and hydraulic networks [2, Th. 2], [3, Sect. 2]. There is much freedom in choosing a suitable three-coloring for a given graph and a given problem. This freedom challenges for creative ideas and makes the theorem elegantly and universally applicable.

Minty utilized the colored branch theorem to iteratively construct solutions for networks containing monotone elements [3]. It has also been used to establish no-gain properties [4], [2, Prop. 1] and source-shifting theorems [5, Th. 6]. In addition, the colored branch theorem is a powerful tool in showing the existence and uniqueness of solutions for several classes of electrical networks [1], [6]. Beside the elegance of incorporating the colored branch theorem in the proof of each of these theorems from circuit theory, it also clearly reveals their topological portion and therefore facilitates a deeper understanding.

Another fundamental result about meshes and cuts in a graph is the orthogonality theorem. The name originates from algebraic topology, where the vector spaces of the meshes and the cuts of a graph prove to be orthogonal. Minty introduced the concept of a graphoid as a generalization of a graph by requiring the meshes and cuts of a graphoid to fulfill a self-dual axiom system that is motivated by both the orthogonality theorem and the colored branch theorem [7].

Our proof of the colored branch theorem incorporates the orthogonality theorem and thereby enhances our understanding of their relation at the graph theoretic foundation of circuit theory. The importance of the colored branch theorem has emerged during the last decades and our demonstration invites to explore the full range of applications even better.

In Section II we recapitulate some basic concepts from graph theory and discuss the orthogonality theorem. A new set theoretic proof of the colored branch theorem is presented in Section III.

II. GRAPH THEORETIC PRELIMINARIES

Definition 1: A triple $(V, E, I)$ is a graph, if $V$ and $E$ are finite, disjoint sets and $I$ satisfies the condition

$$I : E \to V \times V.$$

We call $V$ the set of vertices, $E$ the set of edges and $I$ the incidence mapping. An edge $e$ has end vertices $\hat{v}$ and $\hat{v}$ and is directed from $\hat{v}$ to $\hat{v}$, if $I(e) = (\hat{v}, \hat{v})$.

Definition 2: Let $G = (V, E, I)$ be a graph and $E^+, E^-$ disjoint subsets of $E$ with $E^+ \cup E^- \neq \emptyset$. With $n := |E^+ \cup E^-|$ let $N_n := \{1, 2, \ldots, n\}$ and $N_{n+1} := \{1, 2, \ldots, n+1\}$. The pair $(E^+, E^-)$ is a trail, if there exist mappings

$$\varepsilon : N_n \to E^+ \cup E^-, \quad \nu : N_{n+1} \to V$$

from which $\varepsilon$ is required to be bijective, such that

$$\varepsilon(j) \in E^+ \implies I(\varepsilon(j)) = (\nu(j), \nu(j+1))$$

$$\varepsilon(j) \in E^- \implies I(\varepsilon(j)) = (\nu(j+1), \nu(j))$$

holds for each $j \in N_n$.

A trail, for which $\nu$ is injective, is a path.

A trail with $\nu(1) = \nu(n+1)$ is a mesh.

$(E^+, E^-)$ is an elementary mesh, if $(E^+, E^-)$ is a mesh and there exists no mesh $(\tilde{E}^+, \tilde{E}^-)$ with $\tilde{E}^+ \cup \tilde{E}^- \subset E^+ \cup E^-$. The pair $(E^+, E^-)$ is a cut, if there exists a partition of $V$ into sets $\hat{V}$ and $\hat{V}$ with

$$E^+ = I^{-1}(\hat{V} \times \hat{V})$$

$$E^- = I^{-1}(\hat{V} \times \hat{V})$$

where $I^{-1}$ denotes the inverse relation of $I$.

$(E^+, E^-)$ is an elementary cut, if $(E^+, E^-)$ is a cut and there exists no cut $(\tilde{E}^+, \tilde{E}^-)$ with $\tilde{E}^+ \cup \tilde{E}^- \subset E^+ \cup E^-$.

Proposition 1: Every mesh can be decomposed into pairwise edge-disjoint elementary meshes.

See [8, p. 12] for proof.

Definition 3: Let $G = (V, E, I)$ be a graph and $E^+, E^-$ disjoint subsets of $E$ with $E^+ \cup E^- \neq \emptyset$.

The pair $(E^+, E^-)$ is a cut, if there exists a partition of $V$ into sets $\hat{V}$ and $\hat{V}$ with

$$E^+ = I^{-1}(\hat{V} \times \hat{V})$$

$$E^- = I^{-1}(\hat{V} \times \hat{V})$$

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$$E^- = I^{-1}(\hat{V} \times \hat{V})$$

where $I^{-1}$ denotes the inverse relation of $I$.
Theorem 1 (Orthogonality Theorem): Let \((E^+, E^-)\) be a mesh and \((\hat{E}^+, \hat{E}^-)\) a cut. Then
\[
|E^+ \cap \hat{E}^+| + |E^- \cap \hat{E}^-| = |E^+ \cap \hat{E}^-| + |E^- \cap \hat{E}^+|.
\]

See [9, p. 136] for proof. The orthogonality theorem describes a fundamental relationship between the meshes and the cuts of a graph. Tellegen’s theorem and therefore energy conservation for electrical networks follows from the orthogonality theorem [10, p. 126].

As a conclusion from Theorem 1 we obtain

Proposition 3: A mesh and a cut have an even number of edges in common.

III. THE COLORED BRANCH THEOREM

Definition 5: Let \(G\) be a graph. A triple \((R, G, B)\) is a coloring of the edges of \(G\), if \(R\), \(G\) and \(B\) are pairwise disjoint sets, whose union is equal to the edge set of \(G\), and from which at least one is nonempty.

Following Minty, the edges of the sets \(R\), \(G\) and \(B\) are said to be red, green and blue colored, respectively.

Theorem 2 (Colored Branch Theorem): Let \(G = (V, E, I)\) be a graph and \((R, G, B)\) a coloring of its edges. Let \(G\) be nonempty and \(g\) an element of \(G\).

There exists either an elementary mesh \((E^+, E^-)\) that satisfies \(g \in E^+ \subseteq G \cup R \land E^- \subseteq R\) (1) or an elementary cut \((\hat{E}^+, \hat{E}^-)\) that satisfies \(g \in \hat{E}^+ \subseteq G \cup B \land \hat{E}^- \subseteq B\). (2)

The theorem states, that in a graph, whose edges are colored red, green and blue, respectively, and which has a distinguished green edge \(g\), there exists either a mesh from green and red edges that contains \(g\) or a cut from green and blue edges that contains \(g\), where the green edges are conformly oriented with respect to the mesh and the cut, respectively.

Proof: Let \(\mathcal{M}\) denote the set of elementary meshes of \(G\) and \(\mathcal{C}\) the set of elementary cuts. We define

\[
M : \iff \bigvee_{(E^+, E^-) \in \mathcal{M}} (g \in E^+ \subseteq G \cup R \land E^- \subseteq R)\]
\[
C : \iff \bigvee_{(\hat{E}^+, \hat{E}^-) \in \mathcal{C}} (g \in \hat{E}^+ \subseteq G \cup B \land \hat{E}^- \subseteq B).
\]

To show \(M \lor C\), where \(\lor\) denotes the exclusive-or, we use the equivalence

\[
M \lor C \iff (C \Rightarrow \neg M) \land (\neg M \Rightarrow C)
\]

and show the validity of \(C \Rightarrow \neg M\) and \(\neg M \Rightarrow C\).

Part 1: We prove \(C \Rightarrow \neg M\) by contradiction.

Assume there was an elementary mesh \((E^+, E^-) \in \mathcal{M}\) that satisfies (1) and an elementary cut \((\hat{E}^+, \hat{E}^-) \in \mathcal{C}\) that satisfies (2). Then \(g\) is an element of \(E^+ \cap \hat{E}^+\). By Theorem 1, there exists an edge \(e\), for which \(e \in E^+ \cap \hat{E}^-\) or \(e \in E^- \cap \hat{E}^+\) holds. But it is

\[
E^+ \cap \hat{E}^- \subseteq (G \cup R) \cap B = \emptyset
\]
\[
E^- \cap \hat{E}^+ \subseteq R \cap (G \cup B) = \emptyset.
\]

Both cases lead to a contradiction, therefore the assumption is wrong and the assertion proved.

Part 2: We prove \(\neg M \Rightarrow C\) directly.

We show, how under the premise, that no elementary mesh satisfying (1) exists, an elementary cut satisfying (2) can be constructed.

1) Put

\[
(\hat{v}, \hat{v}) := I(g),
\]
\[
\hat{V} := \{v \in V \mid \text{there exists a path from } \hat{v} \text{ to } v \text{ along arbitrarily oriented red edges and path-conformally oriented green edges} \} \cup \{\hat{v}\}.
\]

2) Partition \(V\) into the sets \(\hat{V}\) and \(\hat{V} := V \setminus \hat{V}\). With

\[
\hat{E}^+ := \hat{I}^{-1}(\hat{V} \times \hat{V})
\]
\[
\hat{E}^- := \hat{I}^{-1}(\hat{V} \times \hat{V})
\]

form the cut \((\hat{E}^+, \hat{E}^-)\) and decompose it into a family of edge-disjoint elementary cuts \(\{(E_i^+, E_i^-)\}_{i \in I}\). Determine \(j \in I\), such that \(g \in E_j^+\) holds, and put

\[
(E^+, E^-) := (E_j^+, E_j^-).
\]

End The elementary cut \((E^+, E^-)\) satisfies (2).

Some supporting comments on the algorithm shall be given. It is \(\hat{v} \in \hat{V}\) and \(\hat{v} \in \hat{V}\), otherwise there was a path from \(\hat{v}\) to \(\hat{v}\) along red edges and path-conformally oriented green edges, which together with \(g\) would form an elementary mesh that contradicts the assumption.

The partition of \(V\) into the sets \(\hat{V}\) and \(\hat{V}\) defines the cut \((\hat{E}^+, \hat{E}^-)\). By construction, \(\hat{E}^+\) contains aside from \(g\) only green and blue colored edges, \(\hat{E}^-\) contains only blue colored edges. Since \((\hat{E}^+, \hat{E}^-)\) is not necessarily an elementary cut, it is decomposed into a family of edge-disjoint elementary cuts. Therefore, \((E^+, E^-)\) with \(g \in E^+\) satisfies (2).

The patient reader is kindly encouraged to derive interesting special cases of the colored branch theorem by coloring all edges of \(G\) green, or by coloring only one edge green and all others red and blue, respectively.

IV. ACKNOWLEDGMENT

The author would like to thank A. Reibiger for his support and many fruitful discussions.

REFERENCES


