Abstract

Let $P_n$ be the class of simple labeled planar graphs with $n$ vertices, and denote by $P_n$ a graph drawn uniformly at random from this set. Basic properties of $P_n$ were first investigated by Denise, Vasconcellos, and Welsh [7]. Since then, the random planar graph has attracted considerable attention, and is nowadays an important and challenging model for evaluating methods that are developed to study properties of random graphs from classes with structural side constraints.

In this paper we study the structure of $P_n$. More precisely, let $b(\ell; P_n)$ be the number of blocks (i.e. maximal biconnected subgraphs) of $P_n$ that contain exactly $\ell$ vertices, and let $lb(P_n)$ be the number of vertices in the largest block of $P_n$. We show that with high probability $P_n$ contains a giant block that includes up to lower order terms $cn$ vertices, where $c \approx 0.959$ is an analytically given constant. Moreover, we show that the second largest block contains only $\Theta(n^{2/3})$ vertices, and prove sharp concentration results for $b(\ell; P_n)$, for all $2 \leq \ell \leq n^{2/3}$ (here $\tilde{\Theta}(.)$ stands for “up to logarithmic factors”).

In fact, we obtain this result as a consequence of a much more general result that we prove in this paper. Let $C$ be a class of labeled connected graphs, and let $C_n$ be a graph drawn uniformly at random from graphs in $C$ that contain exactly $n$ vertices. Under certain assumptions on $C$, and depending on the behavior of the singularity of the generating function enumerating the elements of $C$, $C_n$ belongs with high probability to one of the following three categories, which differ vastly in complexity. $C_n$ either

1. behaves like a random planar graph, i.e. $lb(C_n) \sim cn$, for some analytically given $c = c(C)$, and the second largest block is of order $n^\alpha$, where $1 > \alpha = \alpha(C)$, or
2. $lb(C_n) = O(\log n)$, i.e., all blocks contain at most logarithmically many vertices, or
3. $lb(C_n) = \tilde{\Theta}(n^\alpha)$, for some $\alpha = \alpha(C)$.

Planar graphs belong to category (1). In contrast to that, outerplanar and series-parallel graphs belong to category (2).

Our proofs exploit recent progress in the development of Boltzmann samplers by Duchon, Flajolet, Louchard and Schaeffer [8] and basic tools from modern probability theory.

1 Introduction

Over the last decades the theory of analysis of algorithms was mainly developed from a worst case perspective. While this led to many lines of research with deep and beautiful
results, it also turned out that from a more practical point of view a worst case analysis is often too restrictive, as many real world problems are \textit{NP}-hard and often also \textit{NP}-hard to approximate. This seems to call for an analysis from an average case point of view. As a first step in performing such an analysis for a real world problem one needs to describe a probability distribution on the set of input instances. Even if this point (that is often completely unclear and/or difficult to obtain) could be resolved, it is usually unclear how to proceed from there. This is due to the fact that up to now we are still lacking a powerful machinery that describes how to approach such average case analysis. Even in the case where the inputs come from a somewhat restricted graph class equipped with the uniform distribution we still lack an appropriate easy to use approach.

Over the last years the class of all planar graphs has evolved as a primary example for the development of methods for studying properties of restricted graph classes equipped with the uniform distribution. More precisely, we denote by $\mathcal{P}_n$ the class of all simple labeled planar graphs with $n$ vertices, and use $\mathcal{P}_n$ to denote a graph drawn uniformly at random from this set. Basic properties of $\mathcal{P}_n$ were first investigated by Denise, Vasconcellos, and Welsh [7]. Using a crude counting argument, McDiarmid, Steger, and Welsh [15] showed that a random planar graph in fact has some properties that are quite different from the behaviour of a classical random graph in the Erdős-Rényi model. Namely, they showed that the probability that $\mathcal{P}_n$ is connected is, for $n$ tending to infinity, bounded away from 0 and from 1. (Recall that an Erdős-Rényi graph $G_{n,p}$ satisfies a 0-1 law for all “natural” properties.) While the precise value of this probability is of course given by $\lim_{n \to \infty} \frac{|\mathcal{P}_n|}{|\mathcal{C}_n|}$, where $\mathcal{C}_n$ denotes the class of all labeled connected planar graphs with $n$ vertices, it took quite a while and required deep methods from combinatorial counting and analytic combinatorics to determine the required values asymptotically.

**Theorem 1.1** ([2], [13]). There exist analytically given constants $p, b, \rho_C > 0$ such that

$$|\mathcal{P}_n| \sim pn^{-7/2}\rho_C^{-n}n!$$  and  $$|\mathcal{C}_n| \sim cn^{-7/2}\rho_C^{-n}n!.$$  

While the above result determines precisely the limit of the probability that a random planar graph is connected, the methods that were used in proving this result seem to be restricted to certain kind of questions. For example, even the seemingly “trivial” question of the value of the maximum degree in a random planar graph is extremely hard, if not impossible, to attack by such direct counting arguments.

Using the concept of Boltzmann samplers, cf. Section 3.1, Bernasconi, Panagiotou, and Steger [4, 5] obtained for certain subclasses of planar graphs not only the maximum degree, but also the degree distribution of an element drawn uniformly at random from this class. Their approach relied implicitly on the fact that the classes under consideration have a simple block structure, i.e. they used that all blocks are “small”. In this paper we show that such a property is not true for planar graphs. More precisely, we show that a random planar graph exhibits a block structure whose phenomenology is similar to the component structure of classical Erdős-Rényi random graphs at the threshold $p = 1/n$ [9]. Namely, we show that in a random planar graph there is one giant block containing a linear number of vertices, while the second largest block is of much smaller order, namely $\tilde{\Theta}(n^{2/3})$. Moreover, we show that there are “many” blocks of “small” order.

Recall that the blocks of a graph $G$ are the set of all maximum (induced) subgraphs of $G$ that are biconnected, where with slight abuse of notation we assume that the graph consisting of a single edge is biconnected. For a graph $G$ we denote by $b(\ell; G)$ the number of blocks in
that contain exactly \( k \) vertices. Moreover, let \( lb(G) := \max\{\ell : b(\ell; G) \neq 0\} \), i.e., \( lb(G) \) is the number of vertices in the largest biconnected subgraph of \( G \).

It was shown in [15] that a random planar graph \( P_n \) contains with high probability a giant (connected) component that contains all but a constant number of vertices. We thus restrict our considerations in this paper to studying the block structure of random planar connected graphs. The theorem below summarizes a few “high-level” facts that we can show.

**Theorem 1.2.** Let \( C_n \) be a graph drawn uniformly at random from the set of labeled connected planar graphs with \( n \) vertices. Then the following statements are simultaneously true asymptotically almost surely.

1. The largest block in \( C_n \) contains \( \sim cn \) vertices, where \( c \approx 0.959 \) is analytically given.

2. For \( n^{2/3} \ll \ell < lb(C_n) \) we have \( b(\ell; C_n) = 0 \).

3. The second largest block of \( C_n \) contains at least \( \frac{n^{2/3}}{\log n} \) vertices.

4. For \( 2 \leq \ell \leq \frac{n^{2/3}}{\log n} \) the quantity \( b(\ell; C_n) \) is sharply concentrated around a known value.

While Theorem 1.2 exhibits that random planar graphs have a reasonably complex block structure, this is not the case for some subclasses of planar graphs.

**Theorem 1.3.** Let \( O_n \) be a graph drawn uniformly at random from the set of labeled outerplanar graphs. Then with probability tending to one for \( n \) tending to infinity \( lb(O_n) = O(\log n) \). A similar result holds for the class of series-parallel graphs.

In fact, we obtain Theorems 1.2 and 1.3 as special cases of a much more general theorem. In order to state it we start with a definition.

**Definition 1.4.** Let \( C \) be a class of labeled connected graphs and let \( B \subset C \) be the class of biconnected graphs in \( C \). The class \( C \) is called \((\alpha,\beta)\)-nice if it satisfies the following two properties.

1. Let \( C \in C \) and \( B \in B \). Then the graph obtained by identifying any vertex of \( C \) with any vertex from \( B \) is in \( C \). Moreover, all graphs in \( C \setminus B \) can be constructed in such a way. Finally, the graph consisting of a single isolated vertex is in \( C \).

2. There exist constants \( c, b > 0, \rho_C, \rho_B > 0 \) such that

\[
|C_n| \sim cn^{-\alpha} \rho_C^{-n} n! \quad \text{and} \quad |B_n| \sim bn^{-\beta} \rho_B^{-n} n!.
\]  

Clearly, many natural graph classes are nice, as for example outerplanar, series-parallel, or planar graphs, and more generally, classes that are given in terms of excluded minors. Before we state our main theorem let us recall that the exponential generating function (egf) of a graph class \( G \) is defined as \( G(x) = \sum_{n \geq 1} |G_n| \frac{x^n}{n!} \), where \( G_n \) denotes the set of graphs in \( G \) that contain exactly \( n \) vertices. We will denote by \( \rho_G \) the dominant singularity of \( G(x) \). With these definitions at hand we can now state our main theorem.

**Theorem 1.5.** Let \( C \) be a class of labeled connected graphs that is \((\alpha,\beta)\)-nice, where \( \alpha \geq \frac{5}{2} \) and \( \beta \in \mathbb{R} \). Then the following is true asymptotically almost surely for a graph \( C_n \) drawn uniformly at random from \( C_n \).
(i) If $\rho_B B''(\rho_B) > 1$, then $lb(C_n) = \mathcal{O}(\log n)$.

(ii) If $\rho_B B''(\rho_B) < 1$, then $lb(C_n) \sim (1 - \rho_B B''(\rho_B))n$. Moreover, $\beta > 2$ and we have

1. $b(\ell; C_n) = 0$ for all $n^{1/(\beta-2)} \omega_\beta \leq \ell < lb(C_n)$,

2. $b(\ell; C_n) \sim b\ell n$ for all $2 \leq \ell \leq (\frac{n}{\log n})^{1/(\beta-1)}$, where

$$b_\ell = [x^{\ell-1}]B'(x) \cdot \rho_B^{\ell-1} \sim_\ell b\rho_B^{\ell-\beta+1}, \quad (1.2)$$

3. $b(\ell \ldots \delta; C_n) \sim b_{\ell,\delta} n$ for all $\ell \leq (\frac{n}{\log n})^{1/(\beta-2)}$, where

$$b_{\ell,\delta} = \sum_{s=\ell}^{\delta \ell} [x^{s-1}]B'(x) \cdot \rho_B^{s-1} \sim_\ell \frac{b}{\rho_B^{(\beta-2)}} \cdot (1 - \delta^{-\beta+2})^{\ell-\beta+2}. \quad (1.3)$$

(iii) If $\rho_B B''(\rho_B) = 1$, and $\beta \leq 4$, then $lb(C_n) = \Theta(n^{1/(4\beta-1)})$. Otherwise, there is a block with $\Theta(n^{1/(\beta-2)})$ vertices, and $lb(C_n) = \tilde{O}(\sqrt{n})$.

Note that in the above theorem we require $\alpha \geq 5/2$. We will show in Appendix C that this is in fact a very mild restriction. More precisely, we will show that if we have sufficient information about the behavior of the egf $B(x)$ around its singularity $\rho_B$, then the resulting class $C$ is always nice with $\alpha \geq 5/2$.

Theorems 1.2 and 1.3 are an immediate consequence of Theorem 1.5 as the results of Giménez and Noy [13] and Bodirsky, Giménez, Kang, and Noy [6] imply that planar, outer-planar, and series-parallel graphs are nice.

**Related Results** In contrast to random planar graphs, random planar *maps* and random triangulations are well-studied and well-understood objects. A map is a graph together with an embedding on the plane. Gao and Wormald [12] and Bender, Richmond, and Wormald [3] derived very general results about the size of largest components and applied them to a variety of types of planar maps. Among other results, they showed that a random planar map has a linear size 2-connected submap, and that a 2-connected map has a linear size 3-connected core. These results were generalized and strengthened by Banderier, Flajolet, Schaeffer and Soria [1], who determined the exact distribution of the size of largest (multi-)connected components in several kinds of planar maps. Moreover, they discovered in this context a class of universal phenomena that are of the exponential-cubic type, and correspond to distributions that involve the Airy function.

All the above results are based on completely analytic techniques, singularity analysis of generating functions, and saddle point analysis of complex functions. In the present paper the approach is orthogonal to that: we exploit recent progress in the development of sampling algorithms by Duchon, Flajolet, Louchard and Schaeffer [8], and exploit tools from modern probability theory.

**Notation** In this paragraph we fix the notation that will be used in the article. All graph classes considered here consist of labeled graphs, and without loss of generality we will assume that the vertices of an $n$-vertex graph bear the labels $\{1, \ldots, n\}$. Let $G$ be such a class of labeled graphs. We denote by $G^*$ the class of *vertex-rooted* graphs from $G$, i.e. $G^* = \{(G, v) \mid G \in G, v \text{ is a labeled vertex of } G\}$. In the remainder we shall write with slight abuse
of notation \( G^\bullet \in \mathcal{G}^\bullet \)', meaning that \( G^\bullet \) is a graph from \( \mathcal{G} \) with some distinguished vertex. Finally, we denote by \( G' \) the class of derivated graphs from \( \mathcal{G} \): every graph \( G' \in \mathcal{G}' \) is obtained from a graph \( G \in \mathcal{G} \) by removing the label from the vertex with the maximum label in \( G \). So, if \( G \) had \( n \) vertices, then \( G' \) has \( n \) vertices too, but only \( n-1 \) of them bear a label. We denote the unlabeled vertex in \( G' \) as the virtual vertex of \( G' \).

Finally, we denote by \( \mathcal{G}_n \subset \mathcal{G} \) the class of graphs consisting of precisely \( n \) labeled vertices, and for any \( G \in \mathcal{G}_n \) we write \( |G| = n \) (i.e., \( |G| \) is the number of labeled vertices in \( G \)). So, the above discussion implies that \( |G_n| = n|\mathcal{G}_n| \). Finally, we note that the egf’s for \( \mathcal{G}^\bullet \) and \( \mathcal{G}' \) are \( G^\bullet(x) := x\frac{\partial}{\partial x} G(x) \) and \( G'(x) = \frac{\partial}{\partial x} G(x) \).

2 Preliminaries

In our proofs we will often need to bound the probability that certain random variables assume values far away from their expectation. The next lemma states the well-known Chernoff bounds.

**Lemma 2.1.** Let \( X \sim \text{Bin} (n,p) \). For every \( 0 < \varepsilon < 1 \) it holds \( \Pr [X \notin (1 \pm \varepsilon)np] \leq 2e^{-\varepsilon^2 np/3} \).

Similarly, we will make use of bounds for the tails of a Poisson distributed random variable.

**Lemma 2.2.** Let \( X \sim \text{Po} (\mu) \). For every \( 0 < \varepsilon < 1 \) it holds \( \Pr [X \notin (1 \pm \varepsilon)\mu] \leq 2e^{-\varepsilon^2 \mu/3} \).

The next lemma follows by elementary calculations. For completeness a proof is included in the appendix.

**Proposition 2.3.** Let \( n_0 \geq 4 \) and \( t \geq 1 \) be integers. For any \( \beta > 2 \) there is a constant \( C = C(\beta) > 0 \) such that for any integer \( N \geq t n_0 \)

\[
\sum_{(s_1, \ldots, s_t) \in \mathbb{N}^t} \prod_{i=1}^{t} s_i^{-\beta} \leq C^{t-1} \cdot n_0^{-(\beta-1)(t-1)} \cdot N^{-\beta}.
\]

3 Proofs

In this section we prove our main result. As a preparation we design in Section 3.1 a sampling algorithm for graphs from nice classes, and prove some basic properties of it. Then, in Sections 3.2 and 3.3 we exploit these properties and the combinatorial structure of the graphs in nice classes to carry out the proof of the first two statements of Theorem 1.5. Finally, Appendix B deals with the remaining case \( \rho_B B''(\rho_B) = 1 \).

3.1 Boltzmann Sampling

By applying a standard decomposition of a graph into maximal biconnected subgraphs (see e.g. [14]) we obtain the following combinatorial relation for nice classes of graphs.

**Lemma 3.1.** Let \( \mathcal{C} \) be an \((\alpha, \beta)\)-nice class, and let \( \mathcal{B} \subset \mathcal{C} \) be the class of biconnected graphs in \( \mathcal{C} \). Then \( \mathcal{C}^\bullet = \mathcal{Z} \times \text{SET}(\mathcal{B} \circ \mathcal{C}^\bullet) \).
In words, a rooted connected graph from \( \mathcal{C} \) is an unordered collection of rooted biconnected planar graphs, which are merged at their roots, where every vertex different from the root is subsequently be substituted again with a rooted connected planar graph.

A combinatorial relation as given in the above lemma has two important advantages: first, it can be used to obtain equations that relate the generating functions \( C(x) \) and \( B(x) \) enumerating connected and biconnected graphs. In particular, we get that \( C^\bullet(x) = x e^{B'(C^\bullet(x))} \), see e.g. [14]. Moreover, the relation translates immediately to a randomized sampling algorithm that generates rooted graphs according to an appropriate probability measure defined over the whole class \( \mathcal{C}^\bullet \), the so-called Boltzmann model. This framework was introduced by Duchon, Flajolet, Louchard and Schaeffer in [8], and was extended by Fusy [11]. Here we just present the basic ideas of this framework. Let \( \mathcal{G} \) be a class of labeled graphs. In the Boltzmann model of parameter \( x \) we assign to a graph \( \gamma \in \mathcal{G} \) the probability

\[
\mathbb{P}_x[\gamma] = \frac{1}{G(x)} \frac{G^{\prime}(x)}{|\gamma|!},
\]

if the expression above is well-defined. It is straightforward to see that the expected size of an object in \( \mathcal{G} \) under this probability distribution is \( x G^{\prime}(x) G(x) \). A Boltzmann sampler \( \Gamma \mathcal{G}(x) \) for \( \mathcal{G} \) is an algorithm that generates graphs from \( \mathcal{G} \) according to (3.1). In \([8, 11]\) several general procedures which translate common combinatorial construction rules like union, set, etc. into Boltzmann samplers are given. Notice that the probability above only depends on the choice of \( x \) and on the size of \( \gamma \), such that every object of the same size has the same probability of being generated. This means that if we condition on the output being of a certain size \( n \), then the Boltzmann sampler \( \Gamma \mathcal{G}(x) \) is a uniform sampler for the class \( \mathcal{G}_n \).

In the sequel we are going to demonstrate how we can exploit the rules in \([8, 11]\) to obtain a sampler for \( \mathcal{C}^\bullet \). Define

\[
\lambda_\mathcal{C} := B'(C^\bullet(\rho_\mathcal{C})),
\]

and note that this quantity is finite for \((\alpha, \beta)\)-nice classes (cf. Lemma 3.5 below). Moreover, let \( \Gamma B'(x) \) be a Boltzmann sampler for \( \mathcal{B}' \), i.e. \( \Gamma B'(x) \) samples graphs according to the Boltzmann distribution (3.1) with parameter \( x \) for \( \mathcal{B}' \). Note that here \( 0 \leq x \leq C^\bullet(\rho_\mathcal{C}) \) is admissible, as \( \lambda_\mathcal{C} \) is finite. Then the sampler \( \Gamma \mathcal{C}^\bullet \) for \( \mathcal{C}^\bullet \) is given by the following algorithm.

\[
\Gamma \mathcal{C}^\bullet : \quad \gamma \leftarrow \text{a single node } r \\
\quad k \leftarrow \text{Po}(\lambda_\mathcal{C}) \quad (\star) \\
\quad \text{for } j = 1, \ldots, k \\
\quad \quad \gamma' \leftarrow \Gamma B'(C^\bullet(\rho_\mathcal{C})), \text{ discard the labels of } \gamma' \quad (\star\star) \\
\quad \quad \gamma \leftarrow \text{merge } \gamma \text{ and } \gamma' \text{ at their roots} \\
\quad \text{foreach vertex } v \neq r \text{ of } \gamma \\
\quad \quad \gamma_v \leftarrow \Gamma \mathcal{C}^\bullet, \text{ discard the labels of } \gamma_v \\
\quad \text{replace all nodes } v \neq r \text{ of } \gamma \text{ with } \gamma_v \\
\quad \text{return } \gamma, \text{ where the vertices are labeled uniformly at random}
\]

Note that the above algorithm just reverses the decomposition given in Lemma 3.1: it starts with a single vertex, attaches to it a random set of biconnected graphs, and proceeds recursively to substitute every newly generated vertex by a rooted connected graph. The following lemma is an immediate consequence of the compilation rules in \([8, 11]\).

**Lemma 3.2.** Let \( \gamma \in \mathcal{C}^\bullet \), where \( \mathcal{C} \) is an \((\alpha, \beta)\)-nice class. Then \( \Pr[\Gamma \mathcal{C}^\bullet = \gamma] = \frac{\rho_{|\gamma|}}{|\gamma|! \cdot C^\bullet(\rho_\mathcal{C})} \).
Corollary 3.3. If $C$ is an $(\alpha, \beta)$-nice class, then there is a positive constant $\hat{c}$ such that $\Pr \left[ \Gamma C^* \in C_n^* \right] \sim \hat{c}n^{-\alpha+1}$.

For us it will be convenient to follow an approach first used in [4] and replace the sampler $\Gamma C^*$ by a slightly different sampler with the property that the output distributions of both “algorithms” are the same. Observe that $\Gamma C^*$ makes two kinds of random choices: first, when it choses a random value according to a Poisson distribution in the line marked with $(*)$, and second, when it calls the sampler $\Gamma B'$ in the line marked with (**) (as the return value of this sampler is a random graph from $B'$). We adapt the sampler $\Gamma C^*$ by making the random choices in advance, and by providing them to it as part of its input. More precisely, let $K$ be an infinite sequence of non-negative integers, each one chosen independently according to the parameter $C^*(\rho C)$ from $B'$. Then the sampler $\Gamma C^*(K, B')$, which simulates the execution of $\Gamma C^*$ by using the next unused value from the provided lists, generates obviously every graph from $C^*$ with the same probability as $\Gamma C^*$. In the sequel we will therefore assume that the notation $\Gamma C^*$ in fact denotes the sampler $\Gamma C^*(K, B')$, where we often omit the lists $(K, B')$ for ease of notation.

Recall that $b(\ell; G)$ denotes the number of blocks in a graph $G$ that contain exactly $\ell$ vertices. Our first step in proving Theorem 1.5 is the following lemma, which relates facts about the block structure of a graph generated by $\Gamma C^*(K, B')$ to properties of the lists $(K, B')$. As the latter consist of entries that are sampled independently, this lemma will be a key step in our analysis. The proof can be found in the appendix.

Lemma 3.4. Let $K = \{k_1, k_2, \ldots \}$ be an infinite sequence of non-negative integers and let $B' = \{B'_1, B'_2, \ldots \}$ be an infinite sequence of graphs from $B'$. Suppose that $\Gamma C^*(K, B')$ used the first $n$ values in $K$ and the first $m$ graphs in $B'$ to generate a graph $\gamma \in C^*$. Then the following statements are true.

1. $n = |\gamma|$.
2. $m = \sum_{j=1}^{n} k_j$.
3. $m = \sum_{\ell \geq 2} b(\ell; \gamma)$.
4. For any $\ell \geq 2$ we have that $b(\ell; \gamma) = |\{1 \leq i \leq m \mid |B'_i| = \ell - 1\}|$.

Finally, we need a technical statement about the relation of the singularities of $C$ and $B$.

Lemma 3.5. Let $C$ be an $(\alpha, \beta)$-nice class, and let $B \subset C$ be the set of biconnected graphs in $C$. Let $\rho_C$ and $\rho_B$ be the singularities of the egf’s $C(x)$ and $B(x)$.

- If $\rho_B B''(\rho_B) > 1$, then $C^*(\rho_C) < \rho_B$. Moreover, $\rho_C = \tau e^{-B'(\tau)}$, where $\tau$ is the unique solution to $\tau B''(\tau) = 1$.
- If $\rho_B B''(\rho_B) \leq 1$, then $C^*(\rho_C) = \rho_B$. Moreover, $\rho_C = \rho_B e^{-B'(\rho_B)}$.

3.2 Small Blocks

A first application of Lemma 3.4 is the following statement, which provides us with information about the number of blocks in a random graph $C_n$ from a nice class $C$ that are “small”.
Lemma 3.6. Let $0 < \varepsilon < 1$ and let $C$ be an $(\alpha, \beta)$-nice class. Define the quantities

$$b_\ell = [x^{\ell-1}] B'(x) \cdot (C^\bullet(\rho_C))^{\ell-1} \quad \text{and} \quad \ell_0 = \ell_0(n) = \max \{ \ell \mid b_\ell n \geq 50 \varepsilon^{-2} \alpha \log n \}.$$ 

Then we have for all $2 \leq \ell \leq \ell_0$ and sufficiently large $n$

$$\Pr \left[ b(\ell; C_n) \in (1 \pm \varepsilon) b_\ell n \right] \geq 1 - e^{-\frac{\varepsilon^2}{30} b_\ell n}. \quad (3.3)$$

Proof. In the proof we argue via the Boltzmann sampler $\Gamma C^\bullet$ introduced in the previous section. Fix an $\ell$ such that $2 \leq \ell \leq \ell_0$ and let $S_n \subset C_n$ denote the set of labeled graphs in $C$ on $n$ vertices whose number of blocks of size $\ell$ is not in the interval $(1 \pm \varepsilon) b_\ell n$. Using Corollary 3.3 we obtain for large $n$, with room to spare

$$\Pr \left[ C_n \in S_n \right] = \Pr \left[ \Gamma C^\bullet \in S_n \mid \Gamma C^\bullet \in C^\bullet \right] \leq n^a \Pr \left[ \Gamma C^\bullet \in S_n \text{ and } \Gamma C^\bullet \in C^\bullet \right]. \quad (3.4)$$

In order to estimate the last probability we write $S_n = S_n^{(1)} \cup S_n^{(2)}$, where $S_n^{(1)}$ contains all graphs $G$ in $S_n$ that have the property $\sum_{\ell \geq 2} b(\ell; G) \notin (1 \pm \frac{\varepsilon}{3}) \lambda_C n$, and $S_n^{(2)} = S_n \setminus S_n^{(1)}$. In words, the total number of blocks of every graph in $S_n^{(1)}$ is less than $(1 - \frac{\varepsilon}{3}) \lambda_C n$ or greater than $(1 + \frac{\varepsilon}{3}) \lambda_C n$, where $\lambda_C = B'(C^\bullet(\rho_C))$ is the constant defined in (3.2).

By using Lemma 3.4, statements (1)-(3), the event “$\Gamma C^\bullet \in S_n^{(1)}$ and $\Gamma C^\bullet \in C^\bullet$” implies that the sum of $n$ independent variables distributed like Po($\lambda_C$) is not in $(1 \pm \frac{\varepsilon}{3}) \lambda_C n$. But this probability is at most $e^{-\frac{\varepsilon^2}{30} \lambda_C n}$, due to Lemma 2.2.

Moreover, again by Lemma 3.4, Statement (4), the event “$\Gamma C^\bullet \in S_n^{(2)}$ and $\Gamma C^\bullet \in C^\bullet$” implies that a sequence of $(1 \pm \frac{\varepsilon}{3}) \lambda_C n$ independent random graphs, which are drawn according to the Boltzmann distribution with parameter $C^\bullet(\rho_C)$ from $B'$, contains less than $(1 - \varepsilon) b_\ell n$ or more than $(1 + \varepsilon) b_\ell n$ graphs with $\ell - 1$ non-virtual vertices. We estimate the probability for this event as follows. By exploiting (3.1) we obtain that the probability that a random graph from $B'$ contains $\ell - 1$ non-virtual vertices is precisely $t_{\ell} := [x^{\ell-1}] B'(x) \cdot (C^\bullet(\rho_C))^{\ell-1}$.

Hence, by applying the Chernoff bounds we deduce that the number of graphs with $\ell - 1$ non-virtual vertices among $N$ independently drawn random graphs is less than $(1 - \frac{\varepsilon}{3}) t_{\ell} N$ or more than $(1 + \frac{\varepsilon}{3}) t_{\ell} N$ with probability at most $e^{-\frac{\varepsilon^2}{30} t_{\ell} N}$. The proof completes with $N \in (1 \pm \frac{\varepsilon}{3}) \lambda_C n$ (due to $\Gamma C^\bullet \in S_n^{(2)}$) and the assumption $\lambda_C t_{\ell} n = b_\ell n \geq 50 \varepsilon^{-2} \alpha \log n$.

By exploiting Lemma 3.5 we immediately obtain the following corollary, which proves Part (ii).2. of Theorem 1.5.

Corollary 3.7. If $C$ is an $(\alpha, \beta)$-nice class and $\rho_B B''(\rho_B) < 1$, then

$$b_\ell \sim_\ell \rho_B^{-1} \cdot \ell^{-\beta+1} \quad \text{and} \quad \ell_0 \gg \left( \frac{n}{\log^2 n} \right)^{1/(\beta-1)}.$$ 

Proof. The claim follows immediately from $C^\bullet(\rho_C) = \rho_B$ and the assumption on $C$. 

The next lemma provides us with information about the number of blocks in $C_n$ that contain more than $\ell_0$ vertices, where $\ell_0$ is given in the previous lemma.

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Lemma 3.8. Let $C$ be an $(\alpha, \beta)$-nice class such that $\rho B''(\rho_B) < 1$. Define for $\ell \gg 1$ and $\delta > 1$ the quantities

$$b_{\ell, \delta} = \sum_{s=\ell}^{\ell \pm \delta \ell} [x^{s-1}]B'(x) \cdot \rho_B^{s-1} \quad \text{and} \quad \ell_0 = \ell_0(\delta) = \max \{ \ell \mid b_{\ell, \delta} n \geq 50\varepsilon^{-2} \alpha \log n \}.$$ 

Let $0 < \varepsilon < 1$. Then we have for all $1 \leq \ell \leq \ell_0$ and sufficiently large $n$ for a graph $C_n$ drawn uniformly at random from $C_n$

$$\Pr [b(\ell \ldots \ell_0; C_n) \in (1 \pm \varepsilon)b_{\ell, \delta} n] \geq 1 - e^{-\frac{\varepsilon^2}{20}b_{\ell, \delta} n},$$

and

$$b_{\ell, \delta} \sim \ell \frac{b}{\rho_B(\beta - 2)} \cdot (1 - \delta^{-\beta+2})\ell^{-\beta+2}. \quad (3.5)$$

Proof. As the proof is very similar to the proof of Lemma 3.6, we highlight only the important differences and leave the details to the reader.

Let $B'$ be a random graph drawn according to the Boltzmann distribution with parameter $C^* = \rho_B$ (due to $\rho B''(\rho_B) < 1$ and Lemma 3.5) from $B'$. Then we have that $\Pr [\lvert B' \rvert = k] = \frac{x^k B'(x) \rho_B^k}{b_{B'(\rho_B)}}$, and using the asymptotic estimate $\lvert B_n \rvert \sim b n^{-\beta} \rho_B^{-n} n!$ we readily deduce that $\Pr [\lvert B' \rvert = k] \sim_k b \rho_B^{-1} k^{-\beta+1}$. Using this we obtain by direct Euler-MacLaurin summation

$$\frac{B'(\rho_B) \rho_B}{b} \cdot \Pr [\ell \leq \lvert B' \rvert \leq \delta \ell] \sim \ell \sum_{k=\ell}^{\delta \ell} k^{-\beta+1} \sim \ell \frac{\ell^{-\beta+1} + (\delta \ell)^{-\beta+1}}{2} + \int_{\ell}^{\delta \ell} x^{-\beta+1} dx \sim \ell \frac{1}{\beta - 2} (1 - \delta^{-\beta+2})\ell^{-\beta+2}. \quad (3.6)$$

Note that with the assumption on $\ell$ the above probability is $\geq \frac{50\varepsilon^{-2} \alpha \log n}{n}$. Having this fact we can easily replicate the proof of Lemma 3.6 to obtain the claimed statement. Indeed, let $S_n$ be the set of labeled connected planar graphs on $n$ vertices that have at most $(1 + \varepsilon)b_{\ell, \delta} n$ or at least $(1 - \varepsilon)b_{\ell, \delta} n$ blocks of size in the interval $(\ell, \delta \ell)$, and write $S_n = S_{n(1)} \cup S_{n(2)}$, where $S_{n(1)}$ is defined as in Lemma 3.6, and $S_{n(2)} = S_n \setminus S_{n(1)}$. With this notation we can imitate the derivation in (3.4) and estimate the probability for the event “$C^* \in S_{n(1)}'$ and $\Gamma C^* \in C_n$” in exactly the same way as in Lemma 3.6. Finally, the probability for the event “$C^* \in S_{n(2)}'$ and $\Gamma C^* \in C_n$” can be estimated analogously, where we use for the value $t_{\ell}$ in Lemma 3.6 the value on the right-hand side of (3.6), multiplied by $\frac{b}{B'(\rho_B) \rho_B}$. This completes the proof.

By applying Lemmas 3.6 and 3.8 we immediately obtain that the conclusions (ii), 2, and 3. of Theorem 1.5 are true. Next, we show that the first conclusion of Theorem 1.5 is true, i.e., whenever $C$ is such that $\rho B''(\rho_B) > 1$, then a random graph from $C_n$ has blocks of at most logarithmic size.

Lemma 3.9. Let $C$ be a $(\alpha, \beta)$-nice class such that $\rho B''(\rho_B) > 1$. There is a constant $C = C(C) > 0$ such that for a random graph $C_n$ from $C_n$

$$\Pr [\lceil b(C_n) \rceil \geq C \log n] = o(1).$$
Proof. By applying Lemma 3.5 we infer that \( \tau := C^\bullet(\rho_B) < \rho_B \). Note that in this case the probability that a random graph drawn according to the Boltzmann distribution with parameter \( \tau \) from \( B' \) has \( \ell \) vertices is exponentially small in \( \ell \). More precisely, for large \( \ell \) we have for a random graph \( B' \) from \( B' \) that

\[
\Pr \left[ |B'| = \ell \right] = \frac{|B'_{\ell}| \tau^{\ell}}{B'(x) \ell!} = \mathcal{O} \left( \ell^{-\beta+1} \left( \frac{\tau}{\rho_B} \right)^\ell \right).
\]  

(3.7)

In order to prove the statement we simply replicate the proof of Lemma 3.6. In fact, the proof is exactly the same, with the only difference that we use for \( t_\ell \) (in the end of that proof) the value from (3.7). It is then easily seen that there is constant \( C = C(\alpha, \tau) > 0 \), such that whenever \( \ell > C \log n \), the expected number of blocks of size \( \ell \) is smaller than, say, \( n^{-5-\alpha} \). This completes immediately the proof with the first moment method (for any random variable \( X \) that obtains only non-negative values it holds \( \Pr [X > 0] \leq \mathbb{E}[X] \)).

3.3 Large Blocks in \( C_n \)

Recall that \( b(\ell; G) \) denotes the number of blocks in a graph \( G \) that contain exactly \( \ell \) vertices, and that \( lb(G) \) is the maximum number of vertices in a block of \( G \).

Lemma 3.10. Let \( C \) be an \((\alpha, \beta)\)-nice class such that \( \rho_B B''(\rho_B) < 1 \). For sufficiently large \( n \) we have asymptotically almost surely for a graph \( C_n \) drawn uniformly at random from \( C_n \) that \( lb(C_n) \sim cn \), where \( c = 1 - \rho_B B''(\rho_B) \).

Moreover, let \( \omega_n \) be a function satisfying \( \lim_{n \to \infty} \omega_n = \infty \). Then, for all \( n^{1/(\beta-2)} \omega_n \leq \ell < lb(C_n) \) we have \( b(\ell; C_n) = 0 \).

The proof can be found in Appendix D.

References


A Appendix

Proof of Proposition 2.3. The statement is clearly true for \( t = 1 \). For \( t = 2 \) we need to show

\[
\sum_{x=n_0}^{N-n_0} \frac{1}{x^\beta (N-x)^\beta} \leq C \cdot n_0^{\beta+1} \cdot N^{-\beta}.
\]

Observe that

\[
\sum_{x=n_0}^{(N-n_0)/2} \frac{1}{x^\beta (N-x)^\beta} \leq \sum_{x=n_0}^{(N-n_0)/2} x^{-\beta} \cdot \left( \frac{N}{2} \right)^{-\beta} \\
\leq \left( \frac{N}{2} \right)^{-\beta} \cdot \int_{n_0-1}^{\infty} x^{-\beta} dx \leq \left( \frac{N}{2} \right)^{-\beta} \cdot \frac{(n_0/2)^{-\beta+1}}{\beta-1}.
\]

This proves the claim for \( t = 2 \) and \( C := \frac{2^{\beta+2}}{\beta-1} \). For \( t \geq 3 \) we proceed by induction on \( t \). Observe that

\[
\sum \prod_{i=1}^{t} s_i^{-\beta} = \sum_{x=n_0}^{N-(t-1)n_0} x^{-\beta} \sum_{(s_1,\ldots,s_{t-1}) \in N^t : \sum s_i = N-x, s_i \geq n_0} \prod_{i=1}^{t-1} s_i^{-\beta}.
\]

By applying the induction hypothesis for \( t-1 \) to the second sum we get

\[
\sum \prod_{i=1}^{t} s_i^{-\beta} \leq \sum_{x=n_0}^{N-(t-1)n_0} x^{-\beta} \cdot C^{t-2} n_0^{-(\beta-1)(t-2)} (N-x)^{-\beta},
\]

from which we immediately obtain the claim by again applying the induction hypothesis, this time for \( t = 2 \).

Proof of Lemma 3.4. Before we proceed with the details of the proof let us discuss the general “high-level” strategy. Recall how \( \gamma \) is constructed: in the initial call to the sampler a single vertex \( r \) is generated and the value \( k_1 \) is read. This determines the number of biconnected graphs that have \( r \) as a common vertex. Then the \( k_1 \) graphs \( B'_1,\ldots,B'_{k_1} \) are glued together at \( r \). Denote by \( V \) the set of non-virtual vertices in \( B'_1,\ldots,B'_{k_1} \). Finally, the sampler is called recursively for each \( v \in V \), where \( |V| \) graphs \( (\gamma_v)_{v \in V} \) from \( C^* \) are generated. Note that for all \( v \in V \) we have that \( |\gamma_v| < |\gamma| \). This calls for a proof by induction on \( |\gamma| \). Note that all statements are trivially true for the base case \( |\gamma| = 1 \).

First we perform the induction step for (1). Denote for \( v \in V \) by \( n_v \) the number of variables from \( K \) that the sampler used to generate \( \gamma_v \), and note that due to the hypothesis we have \( n_v = |\gamma_v| \). But then, as in the construction of \( \gamma \) the root vertex of \( \gamma_v \) is identified with \( v \) for all \( v \in V \), we have that

\[
n = 1 + \sum_{v \in V} n_v = 1 + \sum_{v \in V} |\gamma_v| = |\gamma|.
\]

To see (2) let \( V = \{v_1,\ldots,v_{|V|}\} \) and assume that the sampler is called recursively first for \( v_1 \), then for \( v_2 \), and so on. This means, by using (1), that there are indexes \( 2 = i_1 \leq i_2 \leq \cdots \leq i_{|V|} \leq i_{|V|+1} = n+1 \) such that \( \gamma_{v_j} \) is constructed by \( \Gamma C^*(K,B') \) by using the
values \((k_s)_{i_j<s<i_{j+1}}\) from \(K\) (note that if \(i_j = i_{j+1}\), then \(\gamma_{v_j}\) is just the graph consisting of a single vertex). By the induction hypothesis the number \(m_j\) of graphs used from \(B'\) to construct \(\gamma_{v_j}\) is equal to \(\sum_{s=i_j}^{i_{j+1}-1} k_s\). Hence, the number of graphs from \(B'\) used in the construction of \(\gamma\)
is
\[
k_1 + \sum_{j=1}^{[V]} m_j = k_1 + \sum_{j=1}^{[V]} \sum_{s=i_j}^{i_{j+1}-1} k_s = \sum_{j=1}^{n} k_j,
\]
which completes the proof of (2).

As (3) follows immediately from (4), it suffices to prove (4). Observe that the total number \(b(\ell; \gamma)\) of blocks with \(\ell\) vertices in \(\gamma\) is the number of graphs among \(B'_{1}, \ldots, B'_{k_1}\) with \(\ell\) vertices (from which exactly \(\ell - 1\) are non-virtual vertices) plus the total number of blocks with \(\ell\) vertices in \(\gamma_v\) for all \(v \in V\). We deduce
\[
b(\ell; \gamma) = \sum_{i=1}^{k_1} b(\ell; B'_i) + \sum_{v \in V} b(\ell; \gamma_v) = \left\{ 1 \leq i \leq k_1 \mid |B'_i| = \ell - 1 \right\} + \sum_{v \in V} b(\ell; \gamma_v),
\]
and the proof is completed with the induction hypothesis on \(b(\ell; \gamma_v)\) for all \(v \in V\) and (2).

**Proof of Lemma 3.5.** We consider first the case \(\rho_B B''(\rho_B) > 1\). As \(B(x)\) has non-negative coefficients it follows that there is a unique \(0 < \tau < \rho_B\) such that \(\tau B''(\tau) = 1\). Using Theorem VII.2 from [10], where we set \(y(x) = C^\star(x)\) and \(\phi(u) = e^{B''(u)}\), we infer that \(\rho_C = \frac{\tau}{\phi'(\tau)}\), and that a full locally convergent expansion of \(C^\star(x)\) exists, starting with
\[
C^\star(x) \approx x \rho_C \tau - \sqrt{\frac{2\phi'(\tau)}{\phi''(\tau)}} \sqrt{\frac{1}{\rho_C} + o \left( \sqrt{\frac{1}{\rho_C}} \right)}.
\]
In particular, we have that \(C^\star(\rho_C) = \tau < \rho_B\). As a side remark we mention that using Theorem VII.2 from [10] we deduce that there is a constant \(g > 0\) such that \(|C^\star_n| \sim gn^{-3/2} \rho_C^{-n} n!\), or equivalently, \(|C^\star_n| \sim gn^{-5/2} \rho_C^{-n} n!\). So in this case we have that \(\alpha = 5/2\) always.

We now consider the case \(\rho_B B''(\rho_B) \leq 1\). Here, there is no \(0 < \tau < \rho_B\) such that \(\tau B''(\tau) = 1\), and it follows that the dominant singularity of \(C^\star(x)\) is determined by the singularity of \(B(x)\). More precisely, we have that \(C^\star(\rho_C) = \rho_B\). Additionally, as the functional inverse of \(C^\star(x)\) is given by \(\psi(u) = \psi(\rho_B) = \rho_B e^{-B''(\rho_B)}\), by the Analytic Inversion Lemma (see e.g. Proposition IV.5 in [10]). In this case we can also obtain a full locally convergent expansion of \(C^\star(x)\) in terms of the expansion of \(B(x)\), but this is not necessary for our further proofs. However, it will become important in Appendix C, where generalizations of our main result are presented. We postpone the discussion to that point.

**B The case \(\rho_B B''(\rho_B) = 1\)**

In this section we will prove the last statement of Theorem 1.5. Let \(\mathcal{C}\) be an \((\alpha, \beta)\)-nice class, an let \(C_n\) be a graph drawn uniformly at random from \(C_n\). The proof is essentially the same as the proofs of Lemmas 3.6 and 3.8, with the slight difference that we have to control the error terms for the quantities \(b(\ell; C_n)\) more tightly.
In particular, we obtain that \( C_n \) is such that for any \( 2 \leq \ell \leq \log n \)
\[
\Pr \left[ b(\ell; G_n) \in b_\ell n \pm \sqrt{b_\ell n \log n} \right] \geq 1 - n^{-10},
\]
where \( b(\ell; G_n) = [x^{\ell-1}] B'(x) \cdot \rho_B^{\ell-1} \sim \ell b_\ell \rho_B^{-1} \ell^{-\beta+1} \),
by using \( \frac{\log n}{\sqrt{\log n}} \) for \( \varepsilon \) in the proof of Lemma 3.6. Moreover, by mimicking the proof of Lemma 3.8
we obtain completely analogously for any \( \log n \leq \ell < \left( \frac{n}{\log^2 n} \right)^{1/(\beta-2)} \),
\[
\Pr \left[ b(\ell \ldots \log^3 n \cdot \ell; G_n) \in b'_\ell n \pm \sqrt{b'_\ell n \log n} \right] \geq 1 - n^{-10},
\]
where
\[
b'_\ell = \sum_{s=\ell}^{\log^3 n \cdot \ell} [x^{s-1}] B'(x) \cdot \rho_B^{s-1} \sim \ell \frac{b}{\rho_B (\alpha_B - 1)^{\ell-\beta+2}}.
\]
Note that for such \( \ell \) we always have \( b'_\ell n \gg \log n \). Then, by mimicking the proof of Lemma 3.10
we see that the number of vertices in blocks with \( \ell \leq n^{1/(\beta-2)} \log n \) vertices is at least
\[
\sum_{\ell=2}^{\log n} (\ell - 1) (b_\ell n - \sqrt{b_\ell n \log n}) + \sum_{k=1}^{\log n} \log^{3k+1} n \left( b_{\log^{3k+1} n} n - \sqrt{b_{\log^{3k+1} n} n \log n} \right)
\] \[
\geq n \rho_B B''(\rho_B) - \Theta(\log^4 n) - \tilde{O} \left( \sqrt{n} \left( \sum_{\ell=2}^{\log n} \ell^{-\beta-1} + \sum_{k=1}^{\log n} \log^{3k+2} n \cdot (\log^{3k+1} n)^{-\beta+1} \right) \right)
\] \[
\stackrel{(\beta>3)}{\geq} n - \tilde{O} \left( \sqrt{n} \left( \Theta(1) + \log^4 n \cdot n^{4/3(\beta-2)} \right) \right).
\]
So, whenever \( \beta \leq 4 \), we have that \( 1/2 + \frac{4-\beta}{2(\beta-2)} = \frac{1}{\beta-2} \), which implies that \( C_n \) is a.a.s. such
that at most \( \tilde{O}(n^{1/(\beta-3)}) \) vertices are in blocks with more than \( n^{1/(\beta-2)} \log n \) vertices. This
proves the claim for \( \beta \leq 4 \). On the other hand, if \( \beta > 4 \), then there are at most \( \tilde{O}(\sqrt{n}) \) vertices in blocks that contain more than \( n^{1/(\beta-2)} \log n \) vertices. This finishes the proof.

C An analytic view on nice classes

Suppose that the egf for the biconnected graphs of a class defined through its biconnected
components has a unique finite singularity \( \rho_B > 0 \) and admits a full singular expansion of the form
\[
B(x) = \sum_{k \in \mathbb{Z}} s_k \left( 1 - \frac{x}{\rho_B} \right)^{k/\mu},
\]
(C.1)
where \( \mu \) is an integer \( \geq 2 \). We say that the singularity type of \( B/B(x) \) is \( \alpha_B = k/\mu \), where \( k \) is the smallest integer such that \( \alpha_B \notin \mathbb{N}_0 \). This assumption on \( B(x) \) is rather general,
as it includes singularities arising both from algebraic and as well as from meromorphic
functions. Indeed, all the classes mentioned above satisfy (C.1), and most generating functions
encountered in modern analytic combinatorics are of this type.

In the sequel we will argue that graph classes arising from classes \( B \) having a expansion of
the form (C.1) are \( (\alpha, \beta) \)-nice with \( \alpha \geq 5/2 \). We first consider the case \( \rho_B B''(\rho_B) < 1 \). In this case,
there is no \( 0 < \tau < \rho_B \) such that \( \tau B''(\tau) = 1 \), and it follows that the dominant singularity
of \( C^*(x) \) is determined by the singularity of \( B(x) \). More precisely, we have that \( C^*(\rho_C) = \rho_B \). Additionally, as the functional inverse of \( C^*(x) \) is given by \( \psi(u) = ue^{-B'(u)} \), it follows that \( \rho_C = \psi(\rho_B) = \rho_B e^{-B'(\rho_B)} \), by the Analytic Inversion Lemma (see e.g. Proposition IV.5 in [10]).

Note that as the singularity type of \( B(x) \) is \( \alpha_B \), the singularity type of \( B'(x) \) is \( \alpha_B - 1 \), and similarly, the singularity type of \( B''(x) \) is \( \alpha_B - 2 \). Hence, as \( B''(\rho_B) \) exists, we infer that \( \alpha_B > 2 \), which implies that \( B'(x) \) is of singularity type \( > 1 \). Now, having the singular expansion (C.1) of \( B(x) \) at hand, we can readily derive the singular expansions for \( B'(x) \) and \( B''(x) \), which start with the terms

\[
B'(x) = -\frac{s_\mu}{\rho_B} - \frac{2s_{2\mu}}{\rho_B^2} \left( 1 - \frac{x}{\rho_B} \right) + \ldots \quad \text{and} \quad B''(x) = \frac{2s_{2\mu}}{\rho_B^2} + \ldots ,
\]

where the "\ldots" stand for terms of the form \( \left( 1 - \frac{x}{\rho_B} \right)^\xi \) for \( \xi \neq 0 \), which depend on the expansion of \( B(x) \). But then we have that \( s_\mu \neq 0 \), as otherwise \( B'(\rho_B) = 0 \), which contradicts the positivity of the coefficients of \( B(x) \). Moreover, due to our assumption \( \rho_B B''(\rho_B) < 1 \) we may assume that \( \frac{2s_{2\mu}}{\rho_B^2} - 1 \neq 0 \). Exploiting this information we can derive straightforwardly by functional composition a singular expansion of \( \psi(u) \) at \( u = \rho_B \), starting

\[
\psi(u) = \rho_C + \rho_C \left( \frac{2s_{2\mu}}{\rho_B^2} - 1 \right) \left( 1 - \frac{u}{\rho_B} \right) + \ldots .
\]

Given that \( C^*(x) \) and \( \psi(u) \) are functional inverses, we can determine with the above information by indeterminate coefficients the singular expansion of \( C^*(x) \) at \( x = \rho_C \). A straightforward calculation shows that this expansion is of the same singular type as \( B'(x) \), i.e. \( \alpha_B - 1 > 1 \). Then, by applying the Standard Function Scale Theorem (see e.g. Theorem VI.1 from [10]), we readily deduce that that there is a constant \( c > 0 \) such that

\[
|C_n^*| \sim cn^{-\alpha_B} \rho_C^{-n} n! . \tag{C.2}
\]

This completes the proof for the case \( \rho_B B''(\rho_B) \leq 1 \). The case \( \rho_B B''(\rho_B) \leq 1 \) was already treated in the proof of Lemma 3.5 - we omit the details.

**D Proof of Lemma 3.10**

**Proof.** We will show the statement for a random graph \( C_n^* \) from \( C_n^* \), as for every property of graphs \( S \) that is not affected by the presence of the root vertex \( \Pr[P_n \in S] = \Pr[P_n \in S] \).

Let us study the combinatorial structure of graphs from \( C^* \in \mathcal{C}^* \) before we prove the actual statement. Denote by \( m(C^*) \) the number of blocks in \( C^* \), and by \( B(C^*) = \{B_1, \ldots, B_{m(C^*)} \} \) the set of blocks in \( C^* \). We define a partial ordering on \( B(C^*) \) as follows. Let \( r \) be the root vertex of \( C^* \), and let \( D_1 \subseteq B(C^*) \) be the graphs in \( B(C^*) \) that contain \( r \). We say that the graphs in \( D_1 \) have distance one; more precisely, for all \( B \in D_1 \) we write \( \text{dist}(B) = 1 \). Similarly we define the set \( D_s \) that contains all graphs in \( B(C^*) \setminus \bigcup_{i=1}^{s-1} D_i \) that have a common vertex with a graph in \( D_{s-1} \), and for all \( B \in D_s \) we write \( \text{dist}(B) = s \). With this notation we write for two graphs \( B_i, B_j \in B(C^*) \) that \( B_i \leq B_j \) if and only if \( \text{dist}(B_i) \leq \text{dist}(B_j) \). Moreover, we write \( \text{root}(B_i) \) for the vertex of attachment of \( B_i \) to a graph \( B_j \) with smaller distance than \( B_i \) (i.e., we choose \( j \) such that \( \text{dist}(B_j) < \text{dist}(B_i) \)), and note that this vertex
is unique. In the special case that $B_i \in D_1$ we define root($B_i$) = $r$. Finally, we define the set $B'(C^*) = \{B'_1, \ldots, B'_{m(C^*)}\}$, where for all $1 \leq i \leq m(C^*)$ the graph $B'_i$ is identical with $B_i$, with the difference that root($B_i$) is a virtual vertex, i.e., it bears no label.

The above decomposition of graphs from $C^*$ has an important property that will be useful later. Indeed, we can completely describe any graph $C^* \in C^*$ by providing the set $B'(C^*)$ and a sequence of labels $R = \{r_1, \ldots, r|B'(C^*)|\}$, where some labels may occur more than once, such that for all $1 \leq i \leq |B'(C^*)|$ it holds root(($B'(C^*))_i) = r_i$. Here, in slight abuse of the general convention, we will tacitly assume that the graphs in $B'(C^*)$ partition the vertex set $[n]$ of $C^*$ (instead of having labels 1, 2, \ldots).

Let us assume that $C^*$ has $n$ vertices. The above discussion implies that the total number of distinct vertices in the graphs in $D_1$ is $1 + \sum_{B \in D_1} (|B| - 1)$, as all graphs in $D_1$ share the root $r$ of $C^*$ as a common vertex. Similarly, the total number of distinct vertices in the graphs in $D_1 \cup D_2$ is $1 + \sum_{B \in D_1 \cup D_2} (|B| - 1)$, as for every graph $B \in D_2$ there is exactly one vertex of attachment to a graph in $D_1$, namely root($B$).

By iterating this argument we see that the number $n$ of vertices of $P^*$ satisfies

$$n = 1 + \sum_{B \in B(C^*)} (|B| - 1) = 1 + \sum_{B \in B'(C^*)} |B'|,$$

where the last equality is true due to $|B'_i| = |B_i| - 1$ for all $1 \leq i \leq m(C^*)$. With this observation we can prove the claim of the lemma as follows in two steps. First, we will show that the total number of vertices in blocks of $C^*_n$ that contain more than $n^{1/(\beta - 2)} \omega_n$ vertices is $\sim (1 - \rho_B B''(\rho_B))n = cn$. Then we will show that all these vertices are contained in a single huge block.

In the sequel we will perform the first step. By applying Lemma 3.6 we obtain that $C^*_n$ is a.a.s. such that for $2 \leq \ell \leq \left(\frac{n}{\log^2 n}\right)^{1/(\beta - 1)}$ we have $b(\ell; C^*_n) \sim [x^{\ell - 1}] B'(x) \cdot \rho_B^{\ell - 1}$. Moreover, by applying Lemma 3.8 with $\delta = \log^2 n \cdot \omega_n$ we obtain for $\left(\frac{n}{\log^2 n}\right)^{1/(\beta - 1)} \leq \ell \leq \left(\frac{n}{\log^2 n}\right)^{1/(\beta - 2)}$ that a.a.s. $b(\ell \ldots \delta; C^*_n) \sim \sum_{s=\ell}^{\delta} [x^{s-1}] B'(x) \cdot \rho_B^{s-1}$. Let $L(C^*_n)$ be the set of blocks of $C^*_n$ that contain more than $n^{1/(\beta - 2)} \omega_n$ vertices, and define $L'(C^*_n)$ accordingly. By exploiting (D.1) we obtain for large $n$

$$n \sim \sum_{B \in L'(C^*_n)} (|B| - 1) + \sum_{B \in B(L(C^*_n)) \setminus L'(C^*_n)} (|B| - 1) \sim \sum_{B \in L(C^*_n)} (|B| - 1) + \sum_{\ell=2}^{n^{1/(\beta - 2)} \omega_n} b(\ell; C^*_n)(\ell - 1).$$

Note that with our assumptions on $C^*_n$

$$\sum_{\ell=2}^{n^{1/(\beta - 2)} \omega_n} b(\ell; C^*_n)(\ell - 1) \geq n \cdot \sum_{\ell=2}^{\frac{n}{\log^2 n}^{1/(\beta - 1)}} [x^{\ell - 1}] B'(x) \cdot \rho_B^{\ell-1}(\ell - 1) =: n \left(\rho_B B''(\rho_B) - E\right),$$

where due to $|B_n| \sim bn^{-\beta} \rho_B^{-n} n!$ and $x B''(x) = \sum_{\ell \geq 2} [x^{\ell - 1}] B'(x) x^{\ell - 1}(\ell - 1)$

$$E = \sum_{\ell > \left(\frac{n}{\log^2 n}\right)^{1/(\beta - 1)}} [x^{\ell - 1}] B'(x) \rho_B^{\ell - 1}(\ell - 1) \sim \sum_{\ell > \left(\frac{n}{\log^2 n}\right)^{1/(\beta - 1)}} \frac{b}{\rho_B} \ell^{-\beta + 2} = \Theta\left(\left(\frac{n}{\log^2 n}\right)^{\frac{3-\beta}{\beta - 1}}\right).$$

But this last expression is $o(n^{-c})$, for some $c = c(\beta) > 0$, as we may assume that $\beta > 3$ (otherwise $B''(\rho_B)$ would not exist, which contradicts $\rho_B B''(\rho_B) < 1$). On the other hand,
using our assumption for \( \ell \in \left( \left( \frac{n}{ \log^2 n} \right)^{1/(\beta - 1)}, \left( \frac{n}{ \log^2 n} \right)^{1/(\beta - 2)} \right) \) we obtain

\[
\sum_{\ell=2}^{n^{1/(\beta - 2)}c_{\omega}} b(\ell; C^*_n)(\ell - 1) \leq n \rho_B B''(\rho_B) + n \log^2 n \cdot \omega_n \cdot \sum_{\ell \geq \left( \frac{n}{ \log^2 n} \right)^{1/(\beta - 1)}} [x^{\ell - 1}]B'(x) \cdot \rho_B^{-\ell - 1}(\ell - 1)
= n \left( \rho_B B''(\rho_B) + n \log^2 n \cdot \omega_n \cdot E \right).
\]

As the number of blocks in \( L(C^*_n) \) is \( o(n) \), by combining the above facts we obtain

\[
\frac{cn}{n} = \frac{1 - \rho_B B''(\rho_B)}{n} \sim \sum_{B \in L(C^*_n)} (|B| - 1) = \sum_{B' \in L'(C^*_n)} |B'|. \tag{D.2}
\]

In words, \( C^*_n \) is a.a.s. such that the total number of vertices in blocks that contain more than \( n^{1/(\beta - 2)}c_{\omega} \) vertices is \( \sim cn \).

To complete the proof we show that a.a.s. \( |L'(C^*_n)| = 1 \). Let \( \tilde{C}_n \subset C^*_n \) be the set of graphs from \( C^* \) such that every \( C^* \in \tilde{C}_n \) satisfies \( \sum_{B' \in L'(C^*)} |B'| \sim cn \). We will now partition \( \tilde{C}_n \) as follows. Let \( U \subset [n] \) be a set with \( |U| \sim cn \) vertices. Moreover, denote for a graph \( B' \in B' \) by \( V(B') \) the set of non-virtual vertices in \( B' \), and note that \( |V(B')| = |B'| \). We write \( \tilde{C}_n \) as the disjoint union of the sets \( C_U \), where

\[
C_U = \left\{ C^* \in \tilde{C}_n \mid U = \bigcup_{B' \in L'(C^*)} V(B') \right\}.
\]

In the sequel we are going to partition \( C_U \) into further sets as follows. Let \( B'_1, \ldots, B'_e \) be \( c \) graphs from \( B' \) such that \( |B'_i| \leq n^{1/(\beta - 2)}c_{\omega} \) for all \( 1 \leq i \leq c \). Moreover, the \( (B'_i)_{1 \leq i \leq c} \) are such that \( \cup_{i=1}^{\ell} V(B'_i) = [n] \setminus U \), and such that each label from \( [n] \setminus U \) is used exactly once.

In other words, the \( (B'_i)_{1 \leq i \leq c} \) partition \( [n] \setminus U \) in \( c \) classes. Finally, let \( R = \{ r_1, \ldots, r_c \} \) be a multiset from \([n]\). With this information we partition \( C_U \) in the following way. Define the set

\[
C_U(c; B'_1, \ldots, B'_e, R) = \left\{ C^* \in C_U \mid \left\{ B'_1, \ldots, B'_e \right\} \subset B'/(C^*) \text{ and } \forall 1 \leq i \leq c : \text{root}(B'_i) = r_i \right\}.
\]

Observe that this class contains graphs with heavy restrictions: all blocks with \( \leq n^{1/(\beta - 2)}c_{\omega} \) non-virtual vertices are completely fixed, and it is additionally fixed how they connect to the rest of graph. The sole unspecified part is the set \( U \): this contains an unknown number of blocks such that each of them has \( > n^{1/(\beta - 2)}c_{\omega} \) non-virtual vertices, and it not specified what their roots are, i.e., how they connect to the remaining graph.

Note that the sets \( C_U(c; B'_1, \ldots, B'_e, R) \), for each admissible choice of \( c \), each admissible set of graphs \( B'_1, \ldots, B'_e \), and each admissible multiset \( R \), partition \( C_U \). In the remainder we shall therefore fix the set \( U \), a number \( c \geq 1 \), \( c \) graphs \( B'_1, \ldots, B'_e \) with the properties mentioned above, and the multiset \( R \). For each such fixed choice we are going to show that the number of graphs in \( S := C_U(c; B'_1, \ldots, B'_e, R) \) that have a single block that contains all the vertices in \( U \) is \( (1 - o(1))|S| \). This will complete the proof.

Set \( N := |U| \), and recall that \( N \sim cn \). Let us first count the number \( s_1 \) of graphs in \( S \) that have a single block \( b' \) that contains all the vertices in \( U \). As \( C \) is \((\alpha, \beta)\)-nice, we infer that the number of ways to choose that block is \( \sim bN^{-\beta+1}p_B^{-N-1}N! \). Moreover, let \( 1 \leq w = w(U, c, B'_1, \ldots, B'_e, R) \leq n - N \) be the number of ways to connect \( b' \) to the remaining graph, i.e., to specify the quantity \( \text{root}(b') \). So, we have \( s_1 \sim w \cdot bN^{-\beta+1}p_B^{-N-1}N! \).
Next, let us count the number $s_k$ of graphs in $S$ that have $k \geq 2$ blocks $b'_1, \ldots, b'_k$ that contain all the vertices in $U$, i.e., the non-virtual vertices of those $k$ graphs partition $U$. Again by using that $C$ is $(\alpha, \beta)$-nice we infer that this number is at most

$$\sum_{\sum_{i=1}^{k} s_i = N} \sum_{\forall i: s_i \geq n^{1/(\beta-2)}\omega_n} \left( \begin{array}{c} N \\ s_1, \ldots, s_k \end{array} \right) \cdot \prod_{i=1}^{k} bs_{i}^{-\beta+1} \rho_{B}^{-s_{i} \cdot \rho_{B}^{s_{i} + 1}} \cdot \rho_{B}^{-1} s_{i}! = \left( \prod_{i=1}^{k} \frac{s_{i}^{-\beta+1}}{s_{i}!} \right) \cdot b^{k} \rho_{B}^{-k} N^{k} N!. $$

By applying Proposition 2.3 with $n_0 = n^{1/(\beta-2)}\omega_n$, and where we use $\beta - 1$ instead of $\beta$, we deduce that there is a constant $c > 0$ such that this sum is at most

$$c^{k} \cdot (n^{1/(\beta-2)}\omega_n)^{- (\beta-2) (k-1) N^{-\beta+1} \cdot \rho_{B}^{-N-1} N! \leq c^{k} \cdot n^{-k+1} \cdot \omega_{n}^{-k} \cdot N^{-\beta+1} \cdot \rho_{B}^{-N-1} N!}. \quad (D.3)$$

Moreover, the number of ways to attach $b'_1, \ldots, b'_k$ to the remaining graph, i.e., to specify the quantities $\text{root}(b'_i)_{1 \leq i \leq k}$, is at most $k \cdot w \cdot n^{k-1}$. Hence, by using (D.3) we obtain

$$\sum_{k \geq 2} s_k \leq \sum_{k \geq 2} k \cdot w \cdot n^{k-1} \cdot c^{k} \cdot n^{-k+1} \cdot \omega_{n}^{-k} \cdot N^{-\beta+1} \cdot \rho_{B}^{-N-1} N! \leq w \cdot N^{-\beta+1} \cdot \rho_{B}^{-N-1} N! \cdot \sum_{k \geq 2} \left( \frac{c}{\omega_n} \right)^{k} \leq o(w \cdot N^{-\beta+1} \cdot \rho_{B}^{-N} N!) = o(s_1).$$

The proof is completed.