Coloring the edges of a random graph without a monochromatic giant component

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Abstract

Our goal is to color the edges of a random graph $G_{n,m}$ (a graph drawn uniformly at random from all graphs on $n$ vertices with exactly $m$ edges) with a fixed number $r$ of colors such that no color class induces a component of size $\Omega(n)$ – a so called ‘giant component’. We prove that for every $r \geq 2$ there exists an analytically computable constant $c_r^*$ for which the following holds: For any $c < c_r^*$, with probability $1 - o(1)$ there exists an $r$-edge-coloring of $G_{n rcn}$ in which every monochromatic component has sublinearly many vertices. On the other hand, for any $c > c_r^*$, with probability $1 - o(1)$ every $r$-edge-coloring of $G_{n rcn}$ contains a monochromatic component on linearly many vertices. In other words, we prove that the property in question has a sharp threshold at $m = r c_r^* n$.

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1 Introduction

A well-studied question in the theory of random graphs is the following: How does the component structure of the random graph $G_{n,m}$ (a graph drawn

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uniformly at random from all graphs on $n$ vertices with exactly $m$ edges) depend on $m = m(n)$ in the limit $n \to \infty$? In a seminal 1960 paper, Erdős and Rényi discovered, among many other beautiful results, that an abrupt ‘phase transition’ occurs at $m = n/2$. We say that an event occurs \textit{asymptotically almost surely} (a.a.s.) if it occurs with probability $1 - o(1)$ as $n$ tends to infinity. Throughout, we write $G_{n,m}$ where technically we should write $G_{n,\lfloor m \rfloor}$ (or $G_{n,\lceil m \rceil}$) since $m(n)$ is not necessarily an integer.

**Theorem 1.1 ([7])** For any constant $c > 0$ the following holds.

- If $c < 0.5$, then a.a.s. all components of $G_{n,cn}$ have $O(\log n)$ vertices.
- If $c > 0.5$, then a.a.s. $G_{n,cn}$ contains one component with $\Omega(n)$ vertices, and all other components have $O(\log n)$ vertices.

We can also think of this result in the setting of a random graph process where random edges are inserted one by one into an initially empty graph. Then the theorem states that a.a.s. a linear-sized ‘giant component’ emerges quite precisely at the point where the average degree in the graph is 1. In the past decades, much research has been devoted to more detailed studies of the behaviour of the random graph (process) around this critical point; see [5,10] for an overview of results.

A direction of research that has appeared more recently in the literature is whether the emergence of a giant component can be postponed or accelerated if some freedom of choice is introduced in the setup. The most prominent model of this type is the so-called \textit{Achlioptas process}, in which two random edges are sampled at each step, and one of them has to be selected to be included in the graph. Note that this is an online model in which we are required to select edges \textit{before} we have seen the entire random graph. As it turns out, the emergence of a giant component can be both accelerated or slowed down by a constant factor if appropriate edge selection strategies are used [1,2,4,9,14]. However, no threshold result similar to Theorem 1.1 is known for this or other online scenarios.

Motivated by this line of research, several authors have studied similar questions in offline settings. Bohman, Frieze and Wormald considered the following setup: We are given a random graph $G_{n,2cn}$ and are allowed to discard half of its edges. For which values of $c$ can we achieve that the remaining graph with $cn$ edges does not contain a giant component?

**Theorem 1.2 ([2])** There exists an analytically computable constant $c_1 \approx 0.9792$ such that for any constant $c > 0$ the following holds.

- If $c < c_1$, then a.a.s. $G_{n,2cn}$ has a subgraph with $cn$ edges all components of
which have $O(1)$ vertices.

- If $c > c_1$, then a.a.s. every subgraph of $G_{n,2cn}$ with $cn$ edges contains a component with $\Omega(n)$ vertices.

A similar question was studied by Bohman and Kim: For a random graph $G_{n,2cn}$ and a random pairing of its edges, can we select one edge from each pair to obtain a graph with $cn$ edges that does not contain a giant component? Note that this is the offline setting corresponding to the Achlioptas process mentioned above, and that now our choice of edges is more restricted than in the setting of Theorem 1.2. We denote a random graph with a random edge pairing as $G_{n,m}^2$, and say that a subgraph is an Achlioptas subgraph of $G_{n,m}^2$ if it contains exactly one edge from each pair.

**Theorem 1.3 ([3])** There exists an analytically computable constant $c_2 \approx 0.9768$ such that for any constant $c > 0$ the following holds.

- If $c < c_2$, then a.a.s. $G_{n,2cn}^2$ has an Achlioptas subgraph (with $cn$ edges) all components of which have $O(n^{1-\varepsilon})$ vertices, where $\varepsilon$ is a constant depending on $c$.
- If $c > c_2$, then a.a.s. every Achlioptas subgraph of $G_{n,2cn}^2$ contains a component with $\Omega(n)$ vertices.

2 Our result

In this paper we consider a Ramsey-type setting, in which we are required to color the edges of $G_{n,m}$ (with a fixed number $r$ of available colors) instead of selecting a subset of them. Our goal now is to avoid creating a monochromatic giant component in any of the $r$ color classes. (A question more commonly studied in this setup is whether monochromatic copies of some fixed constant-sized graph $F$ can be avoided – a classical result by Rödl and Ruciński gives the general solution for this problem [12,13].) Note that, by the pigeon-hole principle, Theorem 1.2 yields an upper bound for the case $r = 2$.

**Theorem 2.1** For every integer $r \geq 2$, there exists an analytically computable constant $c^*_r$ such that for any constant $c > 0$ the following holds.

- If $c < c^*_r$, then a.a.s. there exists an $r$-edge-coloring of $G_{n,rcn}$ in which all monochromatic components have $O(n^{1-\varepsilon})$ vertices, where $\varepsilon$ is a constant depending on $c$ and $r$.
- If $c > c^*_r$, then a.a.s. every $r$-edge-coloring of $G_{n,rcn}$ contains a monochromatic component with $\Omega(n)$ vertices.
Numerically, we have $c_2^* \approx 0.897$, $c_3^* \approx 0.959$, $c_4^* \approx 0.980$, and $c_5^* \approx 0.990$.

3 Proof sketch

The constants $c_r^*$ in Theorem 2.1 are (up to a factor of $r$) known from the literature and determine the point where the densest subgraph of $G_{n,rcn}$ has average degree $2r$ or, equivalently, where the $(r+1)$-core of $G_{n,rcn}$ has average degree $2r$ [6,8] (the $k$-core of a graph $G$ is the maximal subgraph of $G$ with minimum degree at least $k$). They also determine the thresholds for the property that the edges of $G_{n,rcn}$ can be oriented in such a way that every vertex has in-degree at most $r$. This last result has applications in load balancing, hashing, and related problems, see [6,8] and references contained therein.

The fact that the maximum average degree over all subgraphs of $G_{n,rcn}$ is exactly $2r$ for $c = c_r^*$ relates to our problem as follows: for $c > c_r^*$, the densest subgraph of $G_{n,rcn}$ has average degree strictly larger than $2r$, which by the pigeon-hole principle implies that there is a connected monochromatic subgraph $H$ that has at least $(1 + \varepsilon)|V(H)|$ many edges for some $\varepsilon > 0$. A lemma from [2] asserts that a.a.s. such a subgraph of $G_{n,rcn}$ is necessarily of linear size, which completes the proof of the upper bound.

For the lower bound, we use a trick borrowed from [3]: for $c < c_r^*$, we generate $G_{n,rcn}$ by generating a slightly larger random graph $G^+ = G_{n,rcn+\delta n}$ (for a suitably small $\delta$) and then deleting $\delta n$ edges uniformly at random. Before we proceed with the deletion step, we use the fact that the ratio of edges to vertices in all subgraphs of $G^+$ is less than $r$. By a result of Picard and Queyranne [11], this implies that we can $r$-color the edge set of $G^+$ in such a way that every monochromatic component is unicyclic (‘the pseudoarboricity of $G^+$ is at most $r$’). Analogously to [3], in the deletion step a.a.s. all large monochromatic components of $G^+$ are decomposed into monochromatic components of sublinear size, which implies the lower bound in Theorem 2.1.

References


