Balanced Online Edge Coloring Games on Random Graphs

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Chapter 1

Introduction

Consider the following one-player game on a graph with $n$ vertices: The Player, henceforth called Painter, starts with the empty graph on $n$ vertices. In each step she gets a single edge, which is drawn uniformly at random from the remaining ones, and which she has to color immediately with one of $r$ available colors. She loses the game as soon as she closes a monochromatic copy of a fixed given graph $F$. Her goal is to survive in this random graph process as long as possible. We call this game 'online graph-avoidance edge coloring game'.

How many edges can Painter color asymptotically if she plays optimally? Is there a threshold phenomenon in the usual sense for the duration of this game? How does an optimal strategy for Painter look like?

This game emerged from the following offline problem: How many colors are needed to color a random graph $G_{n,p}$ that is fully revealed without creating a monochromatic copy of a given graph $F$. This offline question was answered in full generality in \cite{RoedlRucinski} by Roedl and Ruciński, who proved semi-sharp thresholds for arbitrary graphs and an arbitrary number of colors.

The idea to transfer the problem into an online scenario came by Friedgut et al. while trying to prove that the threshold for the offline problem is sharp in case of $F$ being a triangle. In \cite{Friedgut} they analyzed the online triangle-avoidance game for 2 colors, say red and blue, and showed that a simple greedy strategy, i.e., coloring every edge red unless this closes a red triangle, is asymptotically best possible and yields a threshold of $n^{4/3}$ for the duration of the game.

Further investigation was done by Marciniszyn, Spoelhel and Steger in \cite{Marciniszyn} and \cite{Marciniszyn2}.
Chapter 1. Introduction

Definition 1. Let $r \geq 1$ and $F$ be a non-empty graph. We set

$$m_r^2(F) := \max_{H_1, H_2, \ldots, H_r \subseteq F} \frac{\prod_{i=1}^r e(H_i)}{2^n + \sum_{i=1}^r (e(H_i) - 2) \prod_{j=1}^{r-1} e(H_j)}.$$  

Theorem 2 (Marciniszyn/Spoehel/Steger, 2007). Let $F$ be a graph that is not a forest, and let $r \geq 1$. Then the online $F$-avoidance edge-coloring game with $r$ colors has a threshold $N(F,r,n)$ that satisfies

$$N(F,r,n) \geq n^{2-1/m_r^2(F)}.$$  

Moreover, if there exists $F_- \subseteq F$ with $e_F - 1$ edges that satisfies

$$m_2(F_-) \leq m_r^2(F),$$

then we have

$$N(F,2,n) = n^{2-1/m_2(F)}.$$  

Their proof of the upper bound uses a two-round approach in which the condition $m_2(F_-) \leq m_r^2(F)$ is used to show that in the first round (within $N(F,2,n)$ steps) Painter must create many ‘threatening situations’ from which at least one forces Painter to lose in the second round (within $\gg N(F,2,n)$ steps).

For the lower bound proof, they generalize the greedy strategy as follows. First, Painter fixes an arbitrary sequence $H_1, H_2, \ldots, H_r \subseteq F$ of subgraphs that maximizes the term in the definition of $m_r^2(F)$. Then, Painter colors every edge with color $r$ unless this closes a monochromatic copy of $H_r$, in which case the strategy switches to color $r-1$ unless this closes a monochromatic copy of $H_{r-1}$ and so on. One can imagine that this strategy produces very unbalanced colorings, i.e., the sizes of the color classes will differ substantially. In fact, the strategy will color all but $o(1)$ edges with color $r$. This observation motivates the question what happens if we restrict the coloring to be balanced, i.e., the sizes of the color classes differ by at most $1$.

We introduce the following modified game, which we call ‘balanced online graph-avoidance edge coloring game’. Painter again starts with the empty graph on $n$ vertices. In each step she is presented $r$ edges which are drawn uniformly at random from the remaining ones, and which she has to color immediately by using each of the $r$ available colors for exactly one of the edges. She loses as soon as she creates a monochromatic copy of $F$. Her goal is to survive as many steps as possible. Note that this game ensures that the coloring produced by Painter is balanced throughout the graph process.

Is there still a threshold phenomenon for this modified game? Does the value of the threshold change? What does an optimal strategy for this game look like?
This game first appeared in [12] where Marciniszyn, Mitsche and Stojakovic proved thresholds for a variety of graphs, including e.g. cycles, for the case of two colors. In this thesis we extend their results to an arbitrary number of colors and to a larger class of graphs. Our result is similar to a result of Prakash and Spoehel in [15] in which they prove a threshold phenomenon for the duration of the corresponding balanced vertex coloring graph avoidance game for a large class of graphs.

**Definition 3.** Let $F$ be a graph. Then we set

$$m_{2b}(F) := \max_{H \subseteq F} \frac{r(e(H) - 1) + 1}{r(v(H) - 2) + 2}.$$ 

**Theorem 4 (Main Result).** Let $F$ be a non-empty graph and $r \geq 1$ an integer.

(i) There exists a strategy such that Painter a.a.s. survives any $N(n) \ll n^{2-1/m_{2b}(F)}$ steps in the balanced online $F$-avoidance edge coloring game.

(ii) Moreover, if there exists $F_- \subseteq F$ with $e_F - 1$ edges that satisfies

$$\max_{H \subseteq F} \frac{e(H) - 1}{v(H) - 2} \leq m_{2b}(F),$$

then Painter loses a.a.s. within any $N(n) \gg n^{2-1/m_{2b}(F)}$ steps regardless of her strategy, that is, $N(F, r, n)$ in this case is a threshold function for the duration of the balanced online $F$-avoidance edge coloring game.

For a variety of graphs this proves an asymptotic threshold phenomenon for the game. Note that the threshold values indeed differ from those for the standard (unbalanced) game. The condition for the upper bound proof can easily be verified to be satisfied for cycles of arbitrary size and cliques of arbitrary size provided that the number $r$ of colors is large enough. Applying our main result we obtain the following statements.

**Corollary 5.** For $r \geq 1$ and $t \geq 3$, the threshold for the balanced online $C_t$-avoidance edge-coloring game is

$$n^{2 - \frac{r(t-2)+2}{r(t-1)+1}}.$$

**Corollary 6.** For $r \geq 1$ and $t \geq r$, the threshold for the balanced online $K_t$-avoidance edge-coloring game is

$$n^{2 - \frac{r(t-2)+2}{r(t-1)+1}}.$$
Driven by the lack of any non-trivial upper bound in Theorem 4 for some graphs we also show a general upper bound on the duration of the game that applies to arbitrary graphs and an arbitrary number of colors. For this, we extend the idea of a two-round approach into an s-round approach, where each round produces graph structures that are 'one edge closer' to $F$ until the last round finally ensures that a monochromatic of $F$ is closed. The exact statement can be found in Theorem 53 on page 46.

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Chapter 2

Preliminaries

We start by introducing the terminology used in this thesis. This chapter is meant to place the technical basis for later chapters. Readers which feel themselves to be familiar with the presented topics may skip single sections or even the whole chapter. The technical character will be loosened in Chapter 3 which introduces the reader into the main issue of this thesis in a more narrative way.

2.1 Graphs

2.1.1 Basics

Definition 7. An (undirected) graph $G = (V, E)$ on $n \geq 1$ vertices consists of a finite set $V = V(G)$ of vertices that satisfies $|V| = n$ and a set $E = E(G) \subseteq \binom{V}{2}$ of edges.

Note that this definition excludes multiple edges and loops. Throughout this thesis we only consider such graphs, which are often called simple in the literature. An example for such a graph is given in Figure 2.1.

We introduce some useful notations and properties of graphs. We write $v_G$ or $v(G)$ for $|V(G)|$, and similarly $e_G$ or $e(G)$ for $|E(G)|$. Clearly, $0 \leq e_G \leq \binom{v_G}{2}$. A graph with $e_G = 0$ (i.e., $E(G) = \emptyset$) is called empty, and a graph with $e_G = \binom{v_G}{2}$ (i.e., $E(G) = \binom{V}{2}$) is called complete.

We say that two vertices $u, v \in V(G)$ are adjacent (in $G$) if they from an edge, i.e., $\{u, v\} \in E(G)$. The neighborhood $\Gamma(v)$ of a vertex $v$ is the set of all vertices that are adjacent to $v$. Its cardinality is called degree of $v$, denoted by $\deg(v) = |\Gamma(v)|$. If $\deg(v) = 0$, then $v$ is called isolated. By $\Delta(G) := \max_{v \in V(G)} \deg(v)$
we denote the maximum degree in $G$. We write $\Gamma_G(v)$ or $\deg_G(v)$ to clarify which surrounding graph we consider the vertex $v$ to be in. Two edges are called \textit{incident} if they have a common endpoint.

Finally, a graph is called \textit{bipartite} if the vertex set $V$ can be partitioned into two subsets $V_1, V_2$ such that no edge runs between them.

\section*{2.1.2 Graph Colorings}

A \textit{graph coloring} or \textit{vertex coloring} is a function $c : V(G) \to \mathbb{N}$, i.e., the colors are denoted by positive integers. It is called \textit{proper} if for all $u, v \in V(G)$ with $\{u, v\} \in E(G)$ we have $c(u) \neq c(v)$, that is, two adjacent vertices must be colored differently.

Similarly, an \textit{edge coloring} is a function $c : E(G) \to \mathbb{N}$. If no two incident edges of $G$ are colored equally, we speak of a \textit{proper} edge coloring.

\section*{2.1.3 Graph Isomorphisms and Subgraphs}

\textbf{Definition 8.} Let $G_1$ and $G_2$ be graphs. A \textit{graph isomorphism} from $G_1$ to $G_2$ is a bijection $\phi : V(G_1) \to V(G_2)$ such that

$$\{u, v\} \in E(G_1) \iff \{\phi(u), \phi(v)\} \in E(G_2).$$

Two graphs $G_1$ and $G_2$ are isomorphic (denoted by $G_1 \cong G_2$) if there is a graph isomorphism between them.

A graph automorphism of $G$ is a graph isomorphism $\phi : V(G) \to V(G)$.

Intuitively, isomorphic graphs are identical except for the labeling of the vertices. To give some necessary conditions for graph isomorphism, we point out that $G_1$
and $G_2$ can clearly only be isomorphic if $v(G_1) = v(G_2)$ and $e(G_1) = e(G_2)$ and even
\[ |\{v \in V(G_1) \mid \deg_{G_1}(v) = i\}| = |\{v \in V(G_2) \mid \deg_{G_2}(v) = i\}| \]
for all $i \in \mathbb{N}_0$. For an example, let $G$ be the graph illustrated in Figure 2.1.

There are two automorphisms of this graph in total, namely the identity and
\[ \phi : V(G) \to V(G) \text{ with } \phi(1) = 1, \phi(2) = 3, \phi(3) = 2 \text{ and } \phi(4) = 4. \]

**Definition 9.** A graph $H$ is a subgraph of $G$ (or contained in $G$) if it is isomorphic to a graph $\tilde{H} = (V(\tilde{H}), E(\tilde{H}))$ with $V(\tilde{H}) \subseteq V(G)$ and $E(\tilde{H}) \subseteq E(G)$.

$H$ is called a proper subgraph if it is not isomorphic to $G$. It is an induced subgraph, also denoted by $G[V(H)]$, if $\tilde{H}$ contains all possible edges, i.e., all edges of $G$ which run between the vertices of $\tilde{H}$.

### 2.1.4 Graph Properties

A graph property $\mathcal{P}$ is a property which holds for all graphs isomorphic to $G$ if it holds for $G$. E.g., ‘$G$ contains a triangle’ is a graph property, ‘$G$ contains vertex $v_1$’ is not. $\mathcal{P}$ also denotes the family of all graphs which satisfy the property $\mathcal{P}$, allowing us to write ‘$G \in \mathcal{P}$’ as a shorthand for ‘$\mathcal{P}$ holds for $G$’.

$\mathcal{P}$ is called increasing if
\[ H \subseteq G \implies (H \in \mathcal{P} \implies G \in \mathcal{P}) \ . \]

Obviously, ‘$G$ contains a triangle’ is an increasing graph property. Similarly, $\mathcal{P}$ is called decreasing if
\[ G \supseteq H \implies (G \in \mathcal{P} \implies H \in \mathcal{P}) \ . \]

### 2.1.5 Special Graphs

To conclude this section, we introduce some graphs of special interest. Here, uniqueness is only meant up to isomorphisms of course.

The $t$-clique, denoted by $K_t$, is the unique graph on $t$ vertices which has $t(t-1)/2$ edges, i.e., any two vertices are adjacent. It is also called the complete graph on $t$ vertices, as mentioned before.

The cycle of size $t$, denoted by $C_t$, is the unique connected graph on $t$ vertices in which every vertex has degree 2.

The complete bipartite graph with parts of size $t$ and $k$, denoted by $K_{t,k}$, is the unique graph $(V,E)$ where the vertex set $V$ can be partitioned into $V_1$ and $V_2$. 

2.2 Asymptotic Notations

Table 2.1 gives an overview of the Landau symbols, which are used to denote the asymptotic behaviour of sequences (or functions) relative to each other. Note that the usage of ‘\(=\)’ in this context is not symmetric, but has always to be read from left to right.

For \(a_n = o(b_n)\) we often use the equivalent notation \(a_n \ll b_n\), similarly \(a_n \gg b_n\) for \(a_n = \omega(b_n)\). We also denote the equivalence relations \(a_n = (1 + o(1))b_n\) and \(a_n = \Theta(b_n)\) by \(\sim\) and \(\asymp\), respectively. In the latter case we also say that \(a_n\) and \(b_n\) are of the same order of magnitude. Observe that \(\sim\) makes a stronger statement than \(\asymp\) since \(\sim\) ensures that \(\lim_{n \to \infty} a_n/b_n = 1\) while \(\asymp\) only implies that this fraction is bounded both away from zero and from above by a constant.

An event \(\mathcal{E}_n\) is said to hold asymptotically almost surely (a.a.s.) if \(\Pr[\mathcal{E}_n] \to 1\) as \(n \to \infty\). The Landau notations have to be used with care when applied in connection with this expression, as for instance the statement \(X_n = O(1)\) a.a.s.’ combines two asymptotic notions. To avoid any ambiguities, we explicitly redefine the notations we will use for random variables in Table 2.2. For example, our definition of \(\Theta(b(n))\) a.a.s. represents the stronger (and more intuitive) of two possible notions, by some authors denoted as \(\Theta_C(b(n))\) in contrast to \(\Theta_p(b(n))\).

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
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<tbody>
<tr>
<td>(a(n) = O(b(n)))</td>
<td>(\exists C &gt; 0 :</td>
</tr>
<tr>
<td>(a(n) = \Omega(b(n)))</td>
<td>(\exists C &gt; 0 :</td>
</tr>
<tr>
<td>(a(n) = \Theta(b(n)))</td>
<td>(a(n) = O(b(n))) and (a(n) = \Omega(b(n)))</td>
</tr>
<tr>
<td>(a(n) = o(b(n)))</td>
<td>(\forall C &gt; 0 :</td>
</tr>
<tr>
<td>(a(n) = \omega(b(n)))</td>
<td>(\forall C &gt; 0 :</td>
</tr>
</tbody>
</table>

Table 2.1: The Landau symbols

such that \(|V_1| = t\), \(|V_2| = k\) and \(E = \{(v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2\}\), i.e., every vertex in \(V_1\) is adjacent to every vertex in \(V_2\) and there are no more edges.

A tree is a connected graph which contains no cycles. A forest is a graph, every component of which is a tree. A star is a special tree that is of the form \(K_{1,t}\) for some \(t \in \mathbb{N}\).

A matching is a non-empty graph with maximum degree 1, i.e., a union of disjoint \(K_2\)’s and isolated vertices. It is called perfect if it contains no isolated vertices.
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<table>
<thead>
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<td>$X_n = \mathcal{O}(b(n))$ a.a.s.</td>
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</tr>
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</tr>
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<td>$\forall C &gt; 0 :</td>
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</table>

Table 2.2: Probabilistic usage of Landau symbols

2.3 Random Graphs and Thresholds

The two most commonly used models for random graphs are $G_{n,p}$ and $G_{n,m}$. In this section we introduce both of them and explain how they are connected. For the most part our presentation follows [9].

In both models the number of vertices is fixed by the parameter $n$ and edges are included in a certain random way. In the $G_{n,p}$ model each of the possible $\binom{n}{2}$ edges is included with probability $p$, independently from all other edges. The corresponding probability space is denoted by $G_{n,p}$. For $p = 1/2$, one easily verifies that this is the uniform distribution on the set of all labeled graphs on $n$ vertices. The number of edges of $G_{n,p}$ is binomially distributed with parameters $\binom{n}{2}$ and probability $p$.

On the other hand, a random graph $G_{n,m}$ is an element drawn uniformly at random from the set of all labeled graphs on $n$ vertices with exactly $m$ edges. The corresponding probability space is denoted by $G_{n,m}$. The main drawback of this model is the fact that the edges of $G_{n,m}$ do not occur independently from each other, which complicates many calculations.

Fortunately, the two models are equivalent in a certain sense for $n \to \infty$, and we may switch to whichever is more convenient in asymptotic calculations. We will give the details below.

For graph properties $\mathcal{P}$ one is usually interested in

$$\lim_{n \to \infty} \mathbb{P}[G_{n,p} \in \mathcal{P}] \ .$$

When dealing with questions of this type, we allow $p$ to be a function of $n$. It turns out that in most cases this limit is either 1 or 0, depending on whether $p(n)$ grows faster or slower than a certain ‘threshold’ function $p_0(n)$.

This motivates the following key definition:

**Definition 10.** $p_0(n)$ is a 0-1-threshold for $\mathcal{P}$ if

$$\mathbb{P}[G_{n,p} \in \mathcal{P}] \to 0 \text{ for } p \ll p_0 \ ,$$
and
\[ \mathbb{P}[G_{n,p} \in \mathcal{P}] \longrightarrow 1 \text{ for } p \gg p_0 \]
as \( n \to \infty \).

We can make essentially the same definition for the \( G_{n,m} \) model:

**Definition 11.** \( m_0(n) \) is a 0-1-threshold for \( \mathcal{P} \) if
\[ \mathbb{P}[G_{n,m} \in \mathcal{P}] \longrightarrow 0 \text{ for } m \ll m_0, \]
and
\[ \mathbb{P}[G_{n,m} \in \mathcal{P}] \longrightarrow 1 \text{ for } m \gg m_0 \]
as \( n \to \infty \).

Analogously, one defines 1-0-thresholds where the values of the limits are interchanged. In most cases the type of threshold is clear and we thus omit the prefix 0-1 or 1-0, respectively.

Observe that weak thresholds are defined up to the order of magnitude, i.e., for all thresholds \( p_0 \) for some graph property \( \mathcal{P} \) the value \( cp_0 \) for some \( c > 0 \) is also a threshold for \( \mathcal{P} \), and conversely, if \( p_0 \) and \( p_1 \) are thresholds for \( \mathcal{P} \) then we have \( p_0 = (1 + o(1))cp_1 \) for some \( c > 0 \).

The importance of this definitions is emphasized by the next theorem, given in [1]:

**Theorem 12** (Bollobás/Thomason, 1987). For all increasing graph properties there exist 0-1-thresholds \( p_0 \) and \( m_0 \).

Thresholds in the sense of Definitions 10 or 11 are sometimes called weak. For some properties even the multiplicative factor is clearly defined (but may be a function of \( n \)), in which case we speak of a sharp threshold.

**Definition 13.** \( p_0(n) \) is a sharp 0-1-threshold for \( \mathcal{P} \) if for arbitrary \( \varepsilon > 0 \)
\[ \mathbb{P}[G_{n,p} \in \mathcal{P}] \longrightarrow 0 \text{ for } p \leq (1 - \varepsilon)p_0, \]
and
\[ \mathbb{P}[G_{n,p} \in \mathcal{P}] \longrightarrow 1 \text{ for } p \geq (1 + \varepsilon)p_0 \]
as \( n \to \infty \).

Weak thresholds which are not sharp are also called coarse.

Łuczak proved a strong connection between \( G_{n,p} \) and \( G_{n,m} \), namely that in all cases, \( p_0 \) and \( m_0 \) are closely related. A proof can be found in [9], pp. 151 - 158.
Theorem 14 (Luczak, 1987). Let \( \mathcal{P} \) be a graph property. Then \( p_0 \) is a (sharp, coarse) threshold for \( \mathcal{P} \) in the \( G_{n,p} \) model if and only if 
\[
m_0 = \left(\frac{n}{2}\right)p_0 = (1 + o(1))n^2p_0/2 \text{ is a (sharp, coarse) threshold for } \mathcal{P} \text{ in the } G_{n,m} \text{ model.}
\]

The connection between the two is quite intuitive, as \( m_0 \) is the expected number of edges in \( G_{n,p_0} \). When considering weak thresholds we usually drop the factor \( 1/2 \) and simply let \( m_0 = n^2p_0 \).

2.4 The Methods of First and Second Moment

In this section we introduce two elementary probabilistic techniques that can be used to prove threshold phenomena in random graphs. For many graph properties \( \mathcal{P} \) there exists a non-negative integer-valued random variable \( X \) such that \( X \neq 0 \) is equivalent to \( \mathcal{P} \). As a short example consider \( \mathcal{P} = G \) contains a triangle. For every set \( C \subseteq V(G) \) with \( |C| = 3 \) we introduce an indicator variable \( X_C \) that is 1 if and only if \( C \) forms a triangle in \( G \), i.e., \( (C^2) \subseteq E(G) \). By setting
\[
X := \sum_{C \subseteq V(G), |C|=3} X_C
\]
we achieve that \( G \in \mathcal{P} \) equals \( X \neq 0 \). To prove a threshold phenomenon for \( \mathcal{P} \) it then suffices to bound \( P[X \neq 0] \) properly. An easy tool for this task is Markov’s inequality.

Lemma 15 (Markov’s inequality). For every non-negative random variable \( X \geq 0 \), and \( t > 0 \) we have
\[
P[X \geq t] \leq \frac{E[X]}{t}.
\]

Proof. Clearly,
\[
E[X] = P[X < t] \cdot E[X \mid X < t] + P[X \geq t] \cdot E[X \mid X \geq t] \\
\geq 0 + P[X \geq t] \cdot t.
\]

The first moment method is an immediate application of Markov’s inequality.

Corollary 16 (First moment method). Let \( (X_n)_{n \in \mathbb{N}} \) be a sequence of non-negative integer-valued random variables. If \( E[X_n] = o(1) \), then we have
\[
P[X_n = 0] = 1 - o(1).
\]
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Proof. We have
\[ P[X_n = 0] = 1 - P[X_n \geq 1] \leq 1 - \mathbb{E}[X_n] = 1 - o(1). \]

Observe that the first moment method can be applied to all graph properties \( \mathcal{P} \) that – as indicated above – are equivalent to \( X \neq 0 \) for some non-negative integer-valued random variable \( X \). In particular it is a tool to prove that \( \mathcal{P} \) a.a.s. does not hold if \( \mathbb{E}[X_n] = o(1) \).

But what about proving that \( \mathcal{P} \) does a.a.s. hold? This is the issue of the second moment method which is an application of Chebyshev’s inequality.

**Lemma 17 (Chebyshev’s inequality).** For every non-negative random variable \( X \geq 0 \) and \( t \geq 0 \) we have
\[ P[|X - \mathbb{E}[X]| \geq t] \leq \frac{\text{Var}[X]}{t^2}. \]

**Proof.** We set \( Y := (X - \mathbb{E}[X])^2 \) and apply Markov’s inequality to \( Y \). This gives us
\[ P[|X - \mathbb{E}[X]| \geq t] = P[Y \geq t^2] \leq \frac{\mathbb{E}[Y]}{t^2} = \frac{\text{Var}[X]}{t^2}. \]

**Corollary 18 (Second moment method).** Let \( (X_n)_{n \in \mathbb{N}} \) be a sequence of non-negative integer-valued random variables. Let \( \mathbb{E}[X_n] = \omega(1) \) and \( \text{Var}[X_n] = o(\mathbb{E}[X_n]^2) \). Then we have
\[ P[X_n = 0] = o(1). \]
Furthermore, we a.a.s. have
\[ X_n \sim \mathbb{E}[X_n]. \]

**Proof.** We have
\[ P[X_n = 0] \leq P[|X_n - \mathbb{E}[X_n]| \geq \mathbb{E}[X_n]] \leq \frac{\text{Var}[X_n]}{\mathbb{E}[X_n]^2} = o(1). \]
Furthermore, observe that for all \( \varepsilon > 0 \) we have
\[ P[(1 - \varepsilon)\mathbb{E}[X_n] < X_n < (1 + \varepsilon)\mathbb{E}[X_n]] = P[|X_n - \mathbb{E}[X_n]| < \varepsilon \mathbb{E}[X_n]] \]
\[ = 1 - P[|X_n - \mathbb{E}[X_n]| \geq \varepsilon \mathbb{E}[X_n]] \]
\[ \geq 1 - \frac{\text{Var}[X_n]}{(\varepsilon \mathbb{E}[X_n])^2} = 1 - o(1). \]
Chapter 3

Motivation and History

Graph theory has had a long history up to now and brought up many beautiful results while still being a vivid area of today’s research. The basic concept of a graph is very simple so that it can be used to model various problems that appear in practice. There is thus a considerable interest in understanding the behavior of graph-related properties of any kind.

3.1 Chromatic Number and Chromatic Index

One field in graph theory that became of major interest over the years is the study of graph colorings. Within this area, the following easy question gained a lot of interest among researchers, maybe partly because of the fact that its answer is by far not that easy: Given a graph $G$. How many colors are needed for a proper vertex coloring of $G$? Clearly, if there is a proper vertex coloring of $G$ with $r \in \mathbb{N}$ colors, then there is also one with $l$ colors for all $l > r$. This gives rise to the notion of the \textit{chromatic number} $\chi(G)$ that denotes the minimum positive integer $r$ such that there exists a proper $r$-vertex coloring of $G$.

While coloring the vertices comes into most people’s minds first, edge colorings are a priori not of less importance, and will in fact be the main focus of this thesis. Transferring the above question from the vertex case, we ask for the minimum number of colors that allows a proper edge coloring of a given graph $G$. This number is called \textit{chromatic index} and denoted by $\chi'(G)$.

Some simple lower bounds can be obtained very easily, as for example

$$\chi(G) \geq \max\{l : K_l \subseteq G\},$$

which states that the chromatic number is at least as big as the largest clique
contained in $G$. This estimate is an immediate consequence of the observation
that all vertices of a clique have to be colored differently in a proper vertex
coloring. An easy lower bound for the chromatic index is

$$\chi'(G) \geq \Delta(G).$$

It says that the chromatic index is at least the maximum degree in $G$, and
follows from the fact that in a proper edge coloring all edges that meet in a
single vertex have to be colored differently. While it is not that easy to obtain
non-trivial upper bounds, a variety of them could still be found over the years.
The most well-known bound on the chromatic number is probably the following.

**Theorem 19** (Brooks’ Theorem, 1941, [3]). For a graph $G$, we have

$$\chi(G) \leq \Delta(G)$$

if and only if $G$ is not a clique or a cycle of odd length in which case $\chi(G) = 
\Delta(G) + 1$.

For the chromatic index there exists the following surprisingly strong upper
bound.

**Theorem 20** (Vizing’s Theorem, 1964, [17]). For all graphs $G$, we have

$$\chi'(G) \leq \Delta(G) + 1.$$

In particular, this shows that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. Nevertheless, the
problems of finding the chromatic number or the chromatic index of a given
graph $G$ are both $NP$-hard.

### 3.2 Avoiding Fixed Subgraphs in Random Graphs

Another big field in graph theory are random graphs, which were first considered
in 1959 by Paul Erdős and Alfréd Rényi in [5]. Since we are interested in
colorings, a natural question is to ask for the chromatic number of a random
graph $G_{n,p}$, which has been answered quite precisely by now (see [2, 8, 11]).

We want to extend the notion of proper colorings. Observe that a proper vertex
coloring is a coloring that avoids monochromatic copies of $K_2$, the complete
graph on 2 vertices, and a proper edge coloring avoids monochromatic copies of
$K_{1,2}$, sometimes also called ‘cherry’. Looking at the problem of finding a proper
coloring from this point of view suggests the following generalisation: Given a
fixed graph $F$. How many colors are needed to color some graph $G$ such that no
monochromatic copy of $F$ appears? Or, regarding random graphs, how many
colors do we expect to need to color $G_{n,p}$ without creating a monochromatic copy of $F$?

Our first step in answering this question is an investigation of the special case of just one single color: Under what conditions can a random graph $G_{n,p}$ be colored with 1 color such that no monochromatic copy of $F$ appears? Note that this is equivalent to an inspection of the graph property ‘$G$ contains a copy of $F$’, and that there is no difference between vertex colorings and edge colorings. This question was answered by Bollobás in 1981 who proved a weak threshold for that graph property.

Before stating his result we need to introduce the graph parameter $m$.

**Definition 21.** Let $F$ be a graph. Then we set

$$d(F) := \frac{e_F}{v_F}.$$  

**Definition 22.** Let $F$ be a graph. Then we set

$$m(F) := \max_{H \subseteq F} d(H).$$  

Intuitively, $m(F)$ is the edge density of its densest subgraph.

**Definition 23.** A graph $F$ is called balanced, if $m(F) = d(F)$.

Throughout this thesis we will encounter several such graph parameters, all of which define thresholds for some graph properties. All of them will be defined by maximizing (as indicated by the letter ‘$m$’) some density measure (which is $d$ in this case) over all subgraphs of $F$. Upcoming graph parameters and density measures will have indices to indicate their purpose.

We can now state Bollobás’ result, which is sometimes also called ‘small subgraphs theorem’.

**Theorem 24** (Bollobás, 1981). Let $F$ be a graph. Then the threshold for the graph property $\mathcal{P} = \{G_{n,p} \text{ contains a copy of } F\}$ is

$$p_0(n) = n^{-1/m(F)}.$$  

Furthermore, if $p \gg p_0(n)$, the number of copies of $F$ in $G_{n,p}$ behaves asymptotically equally to its expectation.

**Proof.** The proof is a good example for the use of the first and second moment method. As for all threshold theorems, it is split into two parts where one proves the assertion for $p \ll p_0$ and the other for $p \gg p_0$.

We fix an arbitrary $H \subseteq F$ such that $m(F) = d(H)$. By $X_H$ we denote the random variable that counts the number of copies of $H$ in $G_{n,p}$. Clearly, $G_{n,p}$
cannot contain a copy of $F$ if it does not contain a copy of $H$, which gives us $P[G_{n,p} \in \mathcal{P}] \leq P[X_H \neq 0]$. Let $\text{Aut}(H)$ denote the set of automorphisms of $H$. We point out that

$$E[X_H] = \left( \binom{n}{v_H} \right) \cdot \frac{v_H!}{|\text{Aut}(H)|} \cdot p^{\epsilon_H}$$

since on a fixed set of $v_H$ out of the overall $n$ vertices there are exactly $\frac{v_H!}{|\text{Aut}(H)|}$ possible copies of $H$ in total each of which is present in the graph with probability $p^{\epsilon_H}$. Assuming $p \ll p_0$ we obtain

$$E[X_H] = \left( \binom{n}{v_H} \right) \frac{v_H!}{|\text{Aut}(H)|} p^{\epsilon_H} = \Theta(n^{v_H}) \cdot \Theta(1) \cdot p^{\epsilon_H} = \Theta(n^{v_H} p^{\epsilon_H}) \quad \text{as } p \ll p_0 \leq o(n^{v_H} p^{\epsilon_H}).$$

With the first moment method (cf. Corollary 16) we obtain that $G_{n,p}$ a.a.s. contains no copy of $H$ if $p \ll p_0$, and thus also no copy of $F$, i.e., $G_{n,p} \not\in \mathcal{P}$ a.a.s.

We now assume $p \gg p_0$. Then we have

$$E[X_F] \approx n^{v_F} p^{\epsilon_F} \gg n^{v_F} p_0^{\epsilon_F} = n^{v_F} n^{-\epsilon_F/m(F)} \geq n^{v_F} n^{-\epsilon_F/d(F)} = 1.$$  

To apply the second moment method it remains to show $\text{Var}[X_F] = o(E[X_F]^2)$. Note that there are $t := \binom{n}{v_F} \cdot (v_F!/|\text{Aut}(F)|)$ possible copies of $F$ in $G_{n,p}$ in total, which we denote by $F_1, F_2, \ldots, F_t$. For each such copy we introduce an indicator variable $X_i$, $1 \leq i \leq t$ for the event $F_i \subseteq G_{n,p}$. Then we have $X_F = \sum_{i=1}^t X_i$. Observe that for all subgraphs $J \subseteq F$ we have

$$n^{-v_J} p^{\epsilon_J} \ll n^{-v_J} p_0^{\epsilon_J} = n^{-v_J + \epsilon_J/m} \leq n^{-v_J + \epsilon_J/d(J)} = 1.$$  

(3.1)
We obtain
\[
\text{Var}[X_F] = \mathbb{E}[X_F^2] - (\mathbb{E}[X_F])^2
= \sum_{F_i, F_j \subseteq K_n} \mathbb{E}[X_i X_j] - \mathbb{E}[X_i] \mathbb{E}[X_j]
= \sum_{J \subseteq F} \sum_{F_i, F_j \subseteq K_n \atop e_J \geq 1, F_i \cap F_j \neq J} p^{2e_F - e_J} - p^{2e_F} + \sum_{F_i, F_j \subseteq K_n \atop E(F_i) \cap E(F_j) = \emptyset} p^{2e_F} - p^{2e_F}
\]
\[
\leq \sum_{J \subseteq F} \sum_{F_i, F_j \subseteq K_n \atop e_J \geq 1, F_i \cap F_j \neq J} p^{2e_F - e_J}
= \sum_{J \subseteq F} \left( \frac{n}{2v_F - v_J} \right) \cdot \Theta(1) \cdot p^{2e_F - e_J}
= \Theta(1) \sum_{J \subseteq F} \Theta(n^{2v_F - v_J}) p^{2e_F - e_J}
= \Theta(\mathbb{E}[X_F^2]) \sum_{J \subseteq F \atop e_J \geq 1} \Theta(n^{-v_J} p^{-e_J})
\]
\[
\text{or } \Theta(\mathbb{E}[X_F^2]) \sum_{J \subseteq F \atop e_J \geq 1} o(1) = o(\mathbb{E}[X_F^2]) .
\]

With the second moment method (cf. Corollary 18) this shows that $G_{n,p} \in \mathcal{P}$ a.a.s. Even more, it shows that
\[X_F \sim \mathbb{E}[X_F]\]
a.a.s. holds.

In fact, it can be shown that this threshold phenomenon is coarse for all graphs $F$. This fully settles the special case of one color, in which there is clearly no difference between vertex colorings and edge colorings. For the remainder of the thesis we will (with the exception of Section 3.6) focus on edge colorings, while there is of course a corresponding vertex coloring case for all problems.

Throughout this thesis $X_F$ denotes the random variable that counts the number of copies of a graph $F$ in a random graph $G_{n,p}$ or $G_{n,m}$.

### 3.3 Avoiding Monochromatic Copies of a Fixed Subgraph

If we allow more colors it becomes harder to determine when $G_{n,p}$ can be colored without creating a monochromatic copy of $F$. Let us first look at the case $p = 1$. 
i.e., coloring \(K_n\), the complete graph on \(n\) vertices. The following deterministic result by Ramsey deals with this case.

**Theorem 25 (Ramsey, 1930).** For any graph \(F\) and \(r \geq 2\), there exists \(n_0 = n_0(F, r)\) such that every \(r\)-edge coloring of \(K_n\) where \(n \geq n_0\) contains a monochromatic copy of \(F\).

A proof of this theorem can be found in [4]. Since the graph \(G_{n,p}\) is fixed for \(p = 1\) the answer to the question only depends on \(n\) and thus does not address any probabilistic issues. For other values of \(p\) we have to modify the question slightly: For what values of \(p\) does every \(r\)-edge coloring a.a.s. contain a monochromatic copy of \(F\)? In [16] Rödl and Ruciński gave an answer to this question. In order to state their result we introduce another graph parameter.

**Definition 26.** Let \(F\) be a graph. Then we set

\[
d_2(F) := \frac{e_F - 1}{v_F - 2}
\]

if \(v_F \geq 3\). Furthermore, we let \(d_2(K_2) = 1/2\), and \(d_2(F) = 0\) if \(v_F < 3\) and \(F \neq K_2\).

**Definition 27.** Let \(F\) be a graph. Then we set

\[
m_2(F) := \max_{H \subseteq F} d_2(H)
\]

A ‘2’ in the index of a density measure \(d\) or its maximizing version \(m\) will always indicate a relation to an edge coloring problem, whereas a ‘1’ will always indicate a relation to a vertex coloring problem.

**Definition 28.** A graph \(F\) is called \(2\)-balanced if \(m_2(F) = d_2(F)\).

**Theorem 29 (Rödl/Ruciński).** Let \(r \geq 2\) and \(F\) be a graph that is not a star forest. Moreover, let \(\mathcal{P} = \{\text{every } r\text{-edge coloring of } G \text{ contains a monochromatic copy of } F\}\). Then there exist constants \(c = c(F, r)\) and \(C = C(F, r)\) such that

\[
P[G_{n,p} \in \mathcal{P}] \rightarrow 1 \quad \text{for } p \geq Cn^{-1/m_2(F)}
\]

and

\[
P[G_{n,p} \in \mathcal{P}] \rightarrow 0 \quad \text{for } p \leq cn^{-1/m_2(F)}
\]

as \(n \to \infty\).

Note that this statement implies that for almost all graphs \(n^{-1/m_2(F)}\) is a threshold for the graph property \(\mathcal{P}\), and that, quite surprisingly, this threshold does not depend on the number of colors. The exceptional case of \(F\) being a star forest is in fact not that hard to settle: By the pigeonhole principle it follows that
there exists an $r$-edge coloring for a graph $G$ without a monochromatic copy of $K_{1,t}$ if and only if all vertices of $G$ have a degree of at most $r(t-1)+1$. With this observation we can obtain a threshold by applying the small subgraphs theorem (Theorem 24) to $K_{1,r(t-1)+1}$.

**Lemma 30.** Let $r \geq 2$ and $F$ be a forest of stars with maximum degree $t$. Then the threshold for the property $\mathcal{P} = \text{'every } r\text{-edge coloring of } G \text{ contains a monochromatic copy of } F \text{' is}

$$p_0(n) = n^{1/m(K_{1,r(t-1)+1})} = n^{1-r(t-1)+1}.$$  

Note that in contrast to Theorem 29 the threshold for star forests does depend on the number of colors. Moreover, since the case is reduced to the appearance of a fixed subgraph (namely $K_{1,r(t-1)+1}$) in $G_{n,p}$ this threshold is coarse since the threshold in Theorem 24 is. Meanwhile the thresholds stated in Theorem 29 are conjectured to be sharp. With two exceptions this conjecture has been verified in [7] for the class of trees, but is still open for all other classes of graphs.

**Lemma 31** (Friedgut/Krivelevich, 2001). Let $r \geq 2$ and $T$ be a tree which is not a star and in the case $r = 2$ not a $P_3$. Furthermore let $\mathcal{P} = \text{'every } r\text{-edge coloring of } G \text{ contains a monochromatic copy of } T \text{'}. Then there exists a constant $c_0 = c_0(T)$ such that

$$\mathbb{P}[G_{n,p} \in \mathcal{P}] \longrightarrow 1 \quad \text{for } p \geq c_0 n^{-1/m_2(F)} = c_0 n^{-1},$$

and

$$\mathbb{P}[G_{n,p} \in \mathcal{P}] \longrightarrow 0 \quad \text{for } p \leq c_0 n^{-1/m_2(F)} = c_0 n^{-1}.$$  

Since their proof is non-constructive, the actual multiplicative constant is still unknown.

Later in this thesis we will need a counting version of Theorem 29 and Lemma 31 that not only proves the existence of a monochromatic copy, but also ensures a certain number of different monochromatic copies.

**Theorem 32** (Rödl/Ruciński, 2001, [16]). Let $r \geq 1$ and $F$ be a non-empty graph. Then there exist constants $C = C(F,r)$ and $a = a(F,r)$ such that for

$$m \geq C n^{2-1/m_2(F)}$$

the random graph $G_{n,m}$ a.a.s. has the property that every $r$-edge coloring of it contains

$$a n^{\nu_F} (m/n^2)^{\nu_F} = \Omega(\mathbb{E}[X_F])$$

monochromatic copies of $F$. 

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Note that star forests are not excluded in this theorem. Since the number of copies of $F$ does a.a.s. not exceed $(1 + \varepsilon)E[X_F]$ by the small subgraphs theorem (Theorem 24), it implies that the number of monochromatic copies of $F$ is $\Theta(E[X_F])$ a.a.s. Also note that there exists at least one color of which there are $\Theta(E[X_F])$ copies.

### 3.4 The Online Graph-Avoidance Game

All problems which we introduced so far in this chapter were dealing with offline coloring problems, that is, the whole graph was presented and we were then to color it. Friedgut et al. were the first to consider colorings that avoid monochromatic copies of a fixed graph $F$ and that are generated online. This process of producing online-colorings can also be described from a (maybe more intuitive) game-related view:

Consider the following one-player game, called online graph-avoidance edge coloring game, short OEC$(F,r,n)$, in a graph on $n$ vertices with $r$ colors and a fixed graph $F$. The game starts with the empty graph, and in each step a new edge that is drawn uniformly at random from the remaining ones appears in the graph. The player – henceforth called Painter – has to color this edge immediately with one of the $r$ available colors. She loses as soon as she creates a monochromatic copy of $F$, and her goal is to survive as long as possible.

How long do we expect the game to last asymptotically (for $n \to \infty$) if Painter plays optimally? Is there a threshold phenomenon in the usual sense? How does an optimal strategy look like?

We first specify what we mean by threshold phenomenon in this game context.

**Definition 33.** Let $r \geq 2$ and $F$ be a non-empty graph. Then $N_0(n) = N_0(F,r,n)$ is a threshold for OEC$(F,r,n)$ if

- there exists a strategy such that for any $N(n) < N_0(n)$ we have
  $$\lim_{n \to \infty} P[\text{Painter survives } N(n) \text{ steps of OEC}(F,r,n)] \to 1,$$
  and
- regardless of Painter’s strategy we have for any $N(n) \gg N_0(n)$ that
  $$\lim_{n \to \infty} P[\text{Painter survives } N(n) \text{ steps of OEC}(F,r,n)] \to 0.$$

In [6], Friedgut et al. now proved that for two colors, say red and blue, and $F$ being a triangle a simple greedy strategy – coloring every edge red unless this closes a red triangle in which case the strategy colors the edge blue – ensures that Painter survives the appearance of the first $o(n^{1/3})$ edges a.a.s. Moreover, they proved that any strategy will a.a.s. fail to color $\omega(n^{1/3})$ edges.
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Theorem 3.4 (Friedgut et al., 2003). The threshold for the online triangle-avoidance edge coloring game with 2 colors is

\[ N(n) = n^{4/3} \]

To prove that a.a.s. \( \omega(n^{4/3}) \) cannot be colored by any strategy they use a two-round approach, splitting the game into coloring the first \( N(n)/2 \) and the last \( N(n)/2 \) edges. It can be shown that Painter must create many ‘almost completed’ triangles (where only one edge is missing, i.e., \( K_{1,2} \)'s) in one of the two colors, say red, in the first round. This also creates many ‘dangerous’ edges that must be colored blue if they appear in the second round. One can prove that Painter will be presented a triangle in the second round that entirely consists of such dangerous edges, forcing her to lose the game.

To avoid any confusion later on we explicitly point out the difference between steps and rounds. A step is the process of presenting Painter a single edge which she immediately colors. In contrast to this, rounds consist of many steps (in this case \( N(n)/2 \)) and will be used in proofs of upper bounds for the duration of online edge coloring games.

The above result was generalized by Marciniszyn, Spoelh and Steger in [13] by proving a lower bound for all fixed graphs \( F \) and any number \( r \) of colors and in [14] where an upper bound for a large class of graphs for the case \( r = 2 \) was shown. This leads us to introducing the next graph parameter.

**Definition 3.5.** Let \( r \geq 1 \) and \( F \) be a non-empty graph. Furthermore let \( H_1, H_2, \ldots, H_r \subseteq F \). Then we set

\[
\overline{d}_2(H_1, H_2, \ldots, H_r) := \frac{\prod_{i=1}^{r} e(H_i)}{2 + \sum_{i=1}^{r} (v(H_i) - 2) \prod_{j=1}^{i-1} e(H_j)} .
\]

While looking slightly complicated at first glance, \( \overline{d}_2(H_1, H_2, \ldots, H_r) \) describes exactly the edge density of the graph illustrated in Figure 3.1 which consists of a central copy of \( H_1 \) and for all \( 1 \leq i \leq r - 1 \) every edge of every copy of \( H_i \) closes a disjoint copy of \( H_{i+1}, H_{i+2}, \ldots, H_r \). We will later give an intuition why this graph is relevant when analyzing OEC(\( F, r, n \)).

**Definition 3.6.** Let \( r \geq 1 \) and \( F \) be a non-empty graph. We set

\[
\overline{m}_2(F) := \max_{H_1, H_2, \ldots, H_r \subseteq F} \overline{d}_2(H_1, H_2, \ldots, H_r) .
\]

Overlining a density measure \( d \) or its maximizing version \( m \) always indicates a relation to an online coloring problem. The following theorem combines the results from [13] and [14].
Let $F$ be a graph that is not a forest, and let $r \geq 1$. Then the online $F$-avoidance edge coloring game with $r$ colors has a threshold $N(F, r, n)$ that satisfies
\[ N(F, r, n) \geq n^{2-1/m_2^r(F)} . \]
Moreover, if there exists $F_- \subseteq F$ with $e_F - 1$ edges that satisfies
\[ m_2(F_-) \leq m_2^r(F) , \]
then we have
\[ N(F, 2, n) = n^{2-1/m_2^r(F)} . \]
In contrast to the offline case, this threshold depends on the number $r$ of colors. One can easily verify that
\[ m(F) = m_1^r(F) < m_2^2(F) < \ldots < m_2^r(F) < \ldots < m_2(F) , \]
and
\[ \lim_{r \to \infty} m_2^r(F) = m_2(F) . \]
Hence the threshold approaches the offline threshold when the number of colors increases.

The proof of the second statement in Theorem 37 also uses the two-round approach introduced earlier, where the existence of such a graph $F_-$ ensures that the first round generates a lot of 'almost completed' monochromatic copies of
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$F$, namely many copies of $F^-$. The strategy that was analyzed for the lower bound proof is the following: Let

$$H_r := \begin{cases} \arg\max_{H \subseteq F} \frac{e_H}{v_H} & \text{if } r = 1, \\ \arg\max_{H \subseteq F} \frac{e_H}{v_H - 2 + \frac{m_{H^-}}{m_{H^-}}(F)} & \text{if } r \geq 2. \end{cases}$$

This defines a sequence $H_1, H_2, \ldots, H_r \subseteq F$ of subgraphs such that $\overline{d}_r(H_1, H_2, \ldots, H_r) = \pi_r^+(F)$. Painter assigns color $r$ to all edges unless this closes a monochromatic copy of $H_r$. In that case, she uses color $r - 1$ unless this closes a monochromatic copy of $H_{r-1}$ and so on. The strategy stops as soon as it creates a copy of $H_1$ in color 1. We might have $H_1 \neq F$, but it turns out that continuing to play until a copy of $F$ in color 1 is closed makes no substantial difference asymptotically. At this point we come back to the graph illustrated in Figure 3.1. In fact, if the edges of this graph appear in a certain order (namely from outer edges to inner edges) the strategy will fail to color this graph structure successfully and lose the game.

One can imagine that Painter uses color $r$ in almost every step, and the generated coloring is highly unbalanced, i.e., the sizes of the color classes differ substantially. Driven by this observation Marciniszyn, Spöhel and Steger suggested to analyze a modified game with some kind of balancedness restriction.

### 3.5 The Balanced Online Graph-Avoidance Game

We modify the online $F$-avoidance edge coloring game such that the edges do not appear one by one, but in sets of $r$ edges that are chosen uniformly at random. Painter has to assign the $r$ colors to the $r$ edges immediately such that each color is used exactly once in every step. We call this game balanced online $F$-avoidance edge coloring game, short $\text{OEC}_b(F, r, n)$. This game is the main focus of this thesis.

We are interested in the question if there is a threshold-behavior for the duration of this modified game as well. If yes, how does this restriction affect the threshold function? How does an optimal strategy look like?

The game was first analyzed by Marciniszyn, Mitsche and Stojaković in [12] for the special case $r = 2$. They presented threshold functions for the duration of the game for certain graphs, e.g. cycles. Their result does in particular not apply to cliques.

In this thesis we extend this result to a larger class of graphs, including cycles and partly cliques, and to an arbitrary number of colors. More precisely, we derive a lower bound on the duration of $\text{OEC}_b(F, r, n)$ for all graphs $F$ and...
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Figure 3.2: An illustration of the graph that motivates $d_{2b}(F)$

$r \geq 1$. Furthermore, we use a two-round approach to prove a matching upper bound under certain conditions on $F$.

**Definition 38.** Let $F$ be a graph. Then we set

$$d_{2b}^r(F) := \frac{1 + r(e_F - 1)}{2 + r(v_F - 2)}.$$  

Observe that $d_{2b}^r(F)$ denotes the edge density of the graph illustrated in Figure 3.2

**Definition 39.** Let $F$ be a graph. Then we set

$$m_{2b}^r(F) := \max_{H \subseteq F} \overline{d}_{2b}(H).$$

The index $b$ indicates the relation to a balanced coloring problem.

**Theorem 40** (Main Result). Let $F$ be a non-empty graph and $r \geq 1$ an integer.

(i) There exists a strategy such that Painter a.a.s. survives any

$$N(n) \ll n^{2-1/m_{2b}^r(F)}$$

steps in $OEC_b(F, r, n)$.

(ii) Moreover, if there exists $F_+ \subseteq F$ with $e_F - 1$ edges that satisfies

$$m_2(F) \leq m_{2b}^r(F),$$

then Painter loses a.a.s. within any

$$N(n) \gg n^{2-1/m_{2b}^r(F)}.$$.

steps regardless of her strategy, that is, $N(F, r, n)$ in this case is a threshold function for the duration of $OEC_b(F, r, n)$.  

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It can be shown (see page 43) that the condition in the second part is satisfied for cycles of arbitrary length, and for cliques of size \(t\) if \(r \geq t\). With this observation we obtain concrete thresholds for these special cases:

**Corollary 41.** For \(r \geq 1\) and \(t \geq 3\), the threshold for the balanced online \(C_t\)-avoidance edge coloring game is
\[
n \left(2 - \frac{t(t-2) + 2}{r(t-1) + 1}\right) - r.
\]

**Corollary 42.** For \(r \geq 1\) and \(t \geq r\), the threshold for the balanced online \(K_t\)-avoidance edge coloring game is
\[
n \left(2 - \frac{t(t-2) + 2}{r(t-1) + 1}\right) - r.
\]

Unfortunately, this does not solve the problem for cliques in full generality due to the condition in the upper bound proof. Addressing this issue, we prove an upper bound for the duration of the balanced online edge coloring game in Chapter 5 which applies to arbitrary graphs and an arbitrary number of colors. This upper bound does not match the lower bound from Theorem 40 and therefore leaves an asymptotic gap. In fact, it can be shown that in many cases the lower bound can be improved by means of a more sophisticated strategy, but it is an open question if the upper bound from Chapter 5 is tight for all graphs.

Since the formalization of this upper bound requires some involved notation, we skip stating it here and just refer to its precise formulation in Theorem 53 on page 46.

### 3.6 The Corresponding Vertex Coloring Problem

We once more return to considering vertex colorings in form of the balanced online graph-avoidance vertex coloring game. This game works slightly different than its corresponding edge coloring version. Painter here starts with a completely empty graph \((\emptyset, \emptyset)\) and consecutively gets \(r\) vertices which are randomly connected to the previous ones (more precisely, each possible edge is connected with probability \(p\), independently of all other edges). Painter has to color these \(r\) vertices immediately using each of the available \(r\) colors exactly once. Her goal is again to avoid monochromatic copies of a fixed graph \(F\) as long as possible.

This game was investigated by Prakash and Spoehel in [13]. The reason for stating their result is that we will transfer some of their proof techniques into our context.

**Definition 43.** Let \(F\) be a graph. Then we set
\[
\overline{d}_{lb}(F) = \frac{r(v_F - 1) + 1}{r(v_F - 1) + 1}.
\]
Figure 3.3: An illustration of the graph that motivates $\overline{d}_{1b}(F)$

Again, $\overline{d}_{1b}(F)$ denotes the edge density of a certain graph structure (see Figure 3.3), which we will motivate after stating the related theorem.

**Definition 44.** Let $F$ be a graph. Then we set

$$m^r_{1b}(F) := \max_{H \subseteq F} d_{1b}(H).$$

**Theorem 45** (Prakash/Spoehel, 2007). Let $r \geq 1$ and $F$ be a non-empty graph. Then there exists a strategy such that Painter a.a.s. survives $N$ steps in the balanced online $F$-avoidance vertex coloring game if

$$p = p(n) \ll n^{-1/m^r_{1b}(F)}.$$  

Moreover, if there exists an induced subgraph $F^0 \subseteq F$ with $v_F - 1$ vertices such that

$$m_1(F^0) := \max_{H \subseteq F^0} \frac{e_H}{v_H - 1} \leq m^r_{1b}(F),$$

then Painter will a.a.s. lose within $N$ steps regardless of her strategy provided that

$$p = p(n) \gg n^{-1/m^r_{1b}(F)}.$$  

One can show that the condition for the second statement of this theorem is satisfied for cycles and cliques of arbitrary size, thereby proving a result that is in some sense stronger than the one in Theorem 40. As for all previously given threshold theorems the condition ensures that in the first round of a two-round approach Painter must generate a lot of ‘threats’ for the second round.

The strategy used in the proof of the first part goes as follows: Choose an arbitrary subgraph $H \subseteq F$ such that $\overline{d}_{1b}(H) = m^r_{1b}(F)$ and try in each step to color the $r$ vertices without creating a monochromatic copy $H$, otherwise
give up. This brings the graph from Figure 3.3 into play. If we imagine the inner vertex to be presented last and all copies of $H$ being monochromatic in a different color, such a graph forces the strategy to give up.
Chapter 4

Bounds on the Duration of \( \text{OEC}_b(F, r, n) \)

The goal of this chapter is to prove the main result as presented in Theorem 4.10. For this, let \( F \) be a non-empty graph and \( r \geq 1 \) an integer. Throughout this section we fix \( F \) and \( r \).

4.1 Formal Setup and Some Elementary Observations

To improve readability we omit floors and ceilings in the following.

4.1.1 \( r \)-matched Graphs

In order to give a convenient and precise description of \( \text{OEC}_b(F, r, n) \) we introduce the concept of an \( r \)-matched graph and an \( r \)-matched random graph where \( r \) is some positive integer. An \( r \)-matched graph \( G' = (V(G'), K(G')) =: (V, K) \) consists of a finite set \( V \) of vertices and a set \( K \subseteq \binom{V}{r} \) of disjoint \( r \)-sets of edges. An illustration of such a graph is given in Figure 4.1. We also use the notation

\[
E(G') := \bigcup_{K \in K} K \subseteq \binom{V}{2}.
\]

For each edge \( e \in E(G') \), we let \( K(e) = K_G(e) \in K \) denote the unique \( r \)-set containing \( e \). For a subset \( E' \subseteq E(G') \), we let

\[
K(E') := K_{G'}(E') := \bigcup_{e \in E'} \{K(e)\} \subseteq K \quad \text{and} \quad V(E') := \bigcup_{e \in E'} e \subseteq V.
\]
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Figure 4.1: A 3-matched graph where the edge-sets of the 3-matching are indicated by different types of dashed lines

For any $K' \subseteq K$, we let

$$E(K') := \bigcup_{K \in K'} K \subseteq E(G^r) \quad \text{and} \quad V(K') = V(E(K')) = \bigcup_{K \in K'} \bigcup_{e \subseteq V} e \subseteq V.$$  

We write $H^r \subseteq G^r$ for $r$-matched graphs $H^r, G^r$ and say `$H^r$ is a subgraph of $G^r$' if $V(H^r) \subseteq V(G^r)$ and $K(H^r) \subseteq K(G^r)$.

By $G^r_{n,m}$ we denote the random $r$-matched graph which is obtained by first generating a normal $G_{n,m}$ and then choosing a random $r$-matching on it. (W.l.o.g. we assume that $m$ is divisible by $r$.)

Furthermore, for an $r$-matched graph $G^r$ let $X_{G^r}$ denote the random variable that counts the number of copies of $G^r$ in a random $r$-matched graph $G^r_{n,m}$. For a family $\mathcal{G}$ of $r$-matched graphs we set $X_\mathcal{G} := \sum_{G^r \in \mathcal{G}} X_{G^r}$.

For later use we state the following lemma that bounds the expected number of appearances of a fixed $r$-matched graph $F^r$ in a random $r$-matched graph.

**Lemma 46.** For every integer $r \geq 1$, and any fixed $r$-matched graph $F^r = (V, K)$ we have

$$\mathbb{E}[X_{F^r}] = O\left(n^{\abs{V} - \abs{K} + 2r \abs{K}}\right).$$

**Proof.** We fix a labeled copy of $F^r$ in $K_n$ and calculate its probability to be present in $G^r_{n,m}$. We split this calculation into two parts, where the first addresses the appearance of the underlying simple graph $(V, E(K))$ in the simple random graph $G_{n,m}$, whereas the second part checks if the $r$-matching on the edges of $F^r$ in $G^r_{n,m}$ conforms with the one of $F^r$.

For the first part we point out that there are \(\binom{n}{2}\) possible results for choosing $m$ edges between $n$ vertices out of which \(\binom{n}{2} - \abs{E(F^r)}\) contain the edges of $F^r$. As for the second part, observe that the probability for a conform
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matching is

\[
\frac{1}{\binom{m-1}{r-1}} \cdot \frac{1}{\binom{m-r-1}{r-1}} \cdots \frac{1}{\binom{m-((|k|-1)r-1)}{r-1}},
\]

which can be seen as verifying for an arbitrary edge \(e \in E(F^r)\) if \(K_{G_{n,m}}(e) = K_{F^r}(e)\) (this probability is exactly \(1/\binom{m}{r-1}\)) and then repeating this verification for the remaining edges until all partitions of \(F^r\) have been checked.

Hence, every possible copy of \(F^r\) is present in \(G_{n,m}^r\) with probability

\[
\frac{\binom{n}{|k|r}}{\binom{m}{|k|r}} \cdot \frac{1}{\binom{m-1}{r-1}} \cdot \frac{1}{\binom{m-r-1}{r-1}} \cdots \frac{1}{\binom{m-((|k|-1)r-1)}{r-1}} \\
\frac{\binom{n-|k|r}{|k|r}}{\binom{m-|k|r}{|k|r}} \cdot \frac{(r-1)!}{(m-1)!} \cdot \frac{(r-1)!}{(m-r-1)!} \cdots \frac{(r-1)!}{(m-((|k|-1)r-1)!}} \\
\frac{(n-|k|r)!}{|k|r!} \cdot (r-1)!^{|k|r} \cdot m(m-r) \cdots (m-((|k|-1)r) \\
\leq \left( \frac{n}{2} - |k|r \right)^{|k|r} \cdot (r-1)!^{|k|r} \cdot m^{|k|r} \\
\leq (n^2/4)^{-r(r-1)!} \cdot m^{|k|r} \\
= \mathcal{O} \left( n^{|k|r/2} r m^{|k|r} \right).
\]

Since, clearly, \(n^{|V|}\) is an upper bound for the number of possible copies of \(F^r\) in \(K_n\), we obtain

\[
\mathbb{E}[X_{F^r}] = \mathcal{O} \left( n^{|V|/2} r m^{|k|r} \right).
\]

4.1.2 Formal Setup

The balanced online graph-avoidance edge-coloring game (short OEC\(b\)(\(F, r, n\))) can now be described formally as a graph process \((G_k^r)_{0 \leq k \leq \binom{n}{2}/r}\) of \(r\)-matched graphs, where \(G_k^r = (V(K_n), \{E_1, E_2, \ldots, E_k\})\) for some partition \(\{E_1, E_2, \ldots, E_{\binom{n}{2}/r}\}\) of \(E(K_n)\) into sets of size \(r\) chosen uniformly at random. In step \(i\) Painter is presented the set \(E_i\), which she has to color immediately without any knowledge about the partition of the remaining edges. She thus consecutively produces balanced colorings of the graph \(G_1^r, G_2^r, \ldots, G_{\binom{n}{2}/r}^r\).
4.1.3 Useful Inequalities and Observations

This section is to provide us with several elementary inequalities and observations that we will need later in the main proof.

**Proposition 47.** For $a, c, C \in \mathbb{R}$ and $b > d > 0$, we have

$$ \frac{a}{b} \leq c \leq \frac{d}{c} \leq C \implies \frac{a + b}{c + d} \leq C \quad \text{and} \quad \frac{a}{b} \geq c \leq \frac{d}{c} \leq C \implies \frac{a - c}{b - d} \geq C .$$

**Proof.** For the first line we point out that the conditions imply $a \leq bC$ and $c \leq dC$ which gives us

$$ a + c \leq (b + d)C .$$

For the second line we note that since $a/b \geq C \geq c/d$, we have

$$ ad \geq bc \quad (4.1) $$

Furthermore,

$$ \frac{a - c}{b - d} = \frac{a - c}{\frac{(a-b)d}{a}} = \frac{a(a-c)}{ab-ad} \geq \frac{a(a-c)}{ab-bc} = \frac{a(a-c)}{b(a-c)} = \frac{a}{b} \geq C .$$

**Corollary 48.** Let $F$ be a non-empty graph. We have

$$ m_{2b}(F) \geq \frac{1}{2} .$$

**Proof.** Let

$$ H_0 := \arg\max_{H \subseteq F} \frac{e_H - 1}{r_H} + 1 .$$

Clearly, $H_0$ does not contain isolated vertices, since removing them would give a graph $H_0$ with $d_{2b}(H_0') > d_{2b}(H_0)$. This yields $e_H/r_H \geq 1/2$. We obtain

$$ m_{2b}(F) = \frac{r(e_H - 1) + 1}{r(v_H - 2) + 2} = \frac{e_H - (1 - 1/r)}{v_H - 2(1 - 1/r)} \overset{\text{Prop. 47}}{\geq} \frac{1}{2} .$$

**Lemma 49.** Let $F$ be a non-empty graph. Then there exists a graph $H \subseteq F$ with $d_2(H) = m_2(F)$ is connected.
Proof. Let $H \subseteq F$ such that $d_2(H) = m_2(F)$ and assume that $H$ is not connected (otherwise the claim follows immediately). Let $C_1$ denote an arbitrary component of $H$. We set $C_2 := (V(H) \setminus V(C_1), E(H) \setminus E(C_1))$, i.e., $C_2$ denotes the union of all other components of $H$. We show that $d_2(C_1) = d_2(H) = m_2(F)$. Note that $H$ cannot contain isolated vertices since removing them increases its edge-density in terms of $d_2$. Hence, we have
\[
\frac{e(H)}{v(H)} \geq \frac{1}{2},
\]
which together with Proposition 47 yields
\[
d_2(H) = \frac{e(H) - 1}{v(H) - 2} = \frac{e(C_1) + e(C_2) - 1}{v(C_1) + v(C_2) - 2} \geq \frac{1}{2}.
\]
Clearly, $C_1, C_2 \subseteq F$ and hence also
\[
d_2(C_1) \leq m_2(F) \quad \text{and} \quad d_2(C_2) \leq m_2(F).
\]
Suppose that $d_2(C_1) < m_2(F)$. Then we have
\[
d_2(H) = \frac{e(C_1) + e(C_2) - 1}{v(C_1) + v(C_2) - 2} \leq \frac{e(C_1) + e(C_2) - 2}{v(C_1) + v(C_2) - 4} = \frac{e(C_1) + e(C_2) - 1}{v(C_1) - 2 + v(C_2) - 2} < m_2(F),
\]
which contradicts $d_2(H) = m_2(F)$. Hence, we have $d_2(C_1) = m_2(F)$.

4.2 The Lower Bound

In this section we prove part (i) of Theorem 40. For this, we need to show that there exists a proper strategy for Painter that lets her survive any $N \ll n^{2-1/m_2(F)}$ steps a.s.

For a graph $H$ we introduce a simple greedy strategy called $H$-avoidance strategy that goes as follows: In step $i$ Painter is presented the edges $E_i$. She chooses an arbitrary permutation to assign the $r$ colors to the $r$ edges that does not produce a monochromatic copy of $H$. If this is not possible she stops. Clearly, if for some $H \subseteq F$ the $H$-avoidance strategy survives $N$ steps for some $N \in \mathbb{N}$, then Painter also avoids a monochromatic copy of $F$ itself.

For the proof of the lower bound we use a strategy called smart greedy strategy which we define to be Painter playing according to the $H$-avoidance strategy for an arbitrary subgraph $H$ defined by
\[
H := \arg\max_{H' \subseteq F} \frac{r(e(H') - 1) + 1}{r(v(H') - 2) + 2}. \tag{4.2}
\]
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The intuition behind the choice of H is the following: We will show that for this choice of H the most likely situation for the H-avoidance strategy to fail (that is, the situation which a.a.s. appears first in the game’s graph process) is the one illustrated in Figure 3.2 where a single edge is presented to Painter that closes a monochromatic copy of H in each color. Ignoring the coloring and just considering the underlying graph \( H^* \), we point out that

\[
d(H^*) = \frac{r(e(H) - 1) + 1}{r(e(H) - 2) + 2}.
\]

Hence, H is a subgraph of F where this structure has maximum edge density and thus appears last in the graph process a.a.s. Or stated differently, the H-avoidance strategy will a.a.s. survive as long as any other H'-avoidance strategy where H' \( \subseteq F \).

For the remainder of the proof we fix H according to (4.2) and use the H-avoidance strategy to prove part (i) of Theorem 40. Let N = N(n) be fixed. Clearly, after N steps we have a random r-matched graph \( G_{n,m} \) on n vertices with exactly m := rN edges. We define a family \( T \) of r-matched graphs which represent ‘traces of a failure’ of the H-avoidance strategy. \( T \) also contains the situation of Figure 3.2 for example.) We show that the H-avoidance strategy can only fail if \( G_N \) contains one of these traces. We will then prove that for m \( \ll n^{2-1/\pi_2(F)} \) a random r-matched graph \( G_{n,m} \) a.a.s. contains no graph of \( T \) which implies that the H-avoidance strategy a.a.s. survives the first N steps. Note that this introduces a ‘static’ view on the problem in the sense that we abstract from the order in which the r-sets of edges appear in the graph process and only look at \( G_N \).

The following definition is illustrated in Figure 4.2. We define for any integer b, 1 \( \leq b \leq r \), the family of (unmatched) graphs

\[
T_b := \{ T = H_1 \cup H_2 \cup \ldots \cup H_r : \ H_1, H_2, \ldots, H_r \cong H \wedge \ \exists e_1 \in E(T) : E(H_i) \cap E(H_j) = \{e_1\}, \ 1 \leq i < j \leq b \wedge \exists e_k \in E(H_k) \setminus \bigcup_{1 \leq i \leq r, i \neq k} E(H_i), \ b + 1 \leq k \leq r \}.
\]

Note that the Figure 4.2 corresponds to the graph in \( T_r \) where the copies are not only edge-, but also vertex-disjoint. We now define the family \( T^r \) of r-matched graphs which describes all possible embeddings of all graphs T \( \in T_b \) for all 1 \( \leq b \leq r \) into an r-matched graph:

\[
T^r := \{ G^r = (V(G^r), K(G^r)) : \exists 1 \leq b \leq r : \exists T \in T_b : \ T \subseteq (V(G^r), E(G^r)) \wedge K(G^r) = K_{G^r}(E(T)) \wedge K(e_1) = K(e_{b+1}) = \ldots = K(e_r) \wedge |E(H_i) \cap K(e_i)| = 1 \ \text{for all} \ 1 \leq i \leq r \}.
\]

(4.3)
Each graph in $T^r$ contains a distinguished edge set $K(e_1)$ (later also denoted by $\tilde{K}$) around which all $r$ copies of $H$ are located. Note that the cardinality of $T^r$ only depends on $F$ and $r$ and is thus constant.

**Lemma 50.** If the $H$-avoidance strategy fails within the first $N$ steps, there exists $T^r \in T^r$ with $T^r \subseteq G_{N}^r$.

**Proof.** Suppose that the $H$-avoidance strategy fails within the first $N$ steps. Then there exists a minimal $k \leq N$ such that the edges $E_k$ cannot be colored without creating a monochromatic copy of $H$. We show that this implies the occurrence of some graph $T^r \in T^r$ in $G_{N}^r$. Assume that $G_{k-1}^r = (V(K_n), \{E_1, \ldots, E_{k-1}\})$ has already been colored successfully, and look at the bipartite graph $B_k$ with $E_k$ as one partition class and the set of available colors $\{1, \ldots, r\}$ as the other partition class, where an edge $e \in E_k$ is connected to a color $s \in \{1, \ldots, r\}$ iff assigning color $s$ to $e$ does not create a monochromatic copy of $H$. Note that the bipartite graph $B_k$ cannot contain a perfect matching since this would allow the $H$-avoidance strategy to color $E_k$ successfully.

Hall’s Theorem (see e.g. [4]) states that in any bipartite graph $G = (V_1 \cup V_2, E)$, a matching of cardinality $|V_1|$ exists if and only if for all subsets $A \subseteq V_1$ the size of their neighborhood $\Gamma(A) \subseteq V_2$ is at least $|A|$. Applied to the situation at hand, there is a set $A \subseteq E_k$ such that its neighborhood $\Gamma(A)$ in $B_k$ has size less than $|A|$ or, equivalently, there exists a set of at least $r - |A| + 1$ colors that are excluded for all of the edges in $A$. Looking at $G_{k-1}^r$, that gives us that each edge $e \in A$ is contained in $r - |A| + 1$ different copies of $H$ which satisfy $|E(H) \cap E_k| = 1$ and pairwise intersect only in $e$ since each of these copies is in a different color. In the remainder of the proof we only consider copies of $H$ in $G_{k-1}^r$ that satisfy $|E(H) \cap E_k| = 1$.

Let $b = r - |A| + 1$ and $e_1, e_2, \ldots, e_{|A|}$ be an arbitrary labeling of the edges in $A$. Moreover, let $H_1, H_2, \ldots, H_b$ denote $b$ copies of $H$ whose edges pairwise intersect in $e_1$, and let $H_{b+1}, \ldots, H_r$ denote copies of $H$ such that each edge in $A \setminus \{e\}$ is covered by exactly one such copy, i.e., $e_i \in E(H_{b+1+i})$ for all $2 \leq i \leq |A|$. Then
clearly, \( T = H_1 \cup H_2 \cup \ldots \cup H_r \in T_b \). Let \( T^r \subset G_{k-1}' \) denote the \( r \)-matched graph that is induced by \( T \) in \( G_{k-1}' \), that is, \( T^r := (V(K_{G_{k-1}'(E_T)}), K_{G_{k-1}'(E_T)}) \).

Since \( K_{G_{k-1}'(e)} = E_k \) for all \( e \in A \), we also have \( T^r \in T' \).

We use the first moment method to prove that a.a.s. \( X_{T^r} = 0 \) provided that \( N \ll n^{2-1/m_{2k}(F)} \), which together with the above lemma yields that a.a.s. the \( H \)-avoidance strategy is successful. For this we will show that if \( N \ll n^{2-1/m_{2k}(F)} \), we have for all \( T^r \in T' \) \( \mathbb{E}[X_{T^r}] = o(1) \). Since the number of \( r \)-matched graphs in \( T' \) is constant as mentioned before, we obtain

\[
\mathbb{E}[X_{T^r}] = \sum_{T^r \in T'} \mathbb{E}[X_{T^r}] = o(1) .
\]

The first moment method then ensures that a.a.s. we have \( X_{T^r} = 0 \). It remains to show the following lemma.

**Lemma 51.** If \( N \ll n^{2-1/m_{2k}(F)} \), we have \( \mathbb{E}[X_{T^r}] = o(1) \) for all \( T^r \in T' \).

**Proof.** Let \( T^r = (V(K), K) \in T' \) with its corresponding parameter \( 1 \leq b \leq r \) and \( K \in K \) being its distinguished \( r \)-set of edges and \( T = H_1 \cup H_2 \cup \ldots \cup H_r \in T_b \) being the underlying unmatched graph holding the \( r \) copies of \( H \) around \( \tilde{K} \). We set \( \{ e_i \} := E(H_i) \cap \tilde{K} \) for all \( 1 \leq i \leq r \) so that \( e_i \) denotes the unique edge of \( H_i \) in \( \tilde{K} \). Note that \( e_1 = e_2 = \ldots = e_b \). For all \( 2 \leq i \leq r \) let

\[
J_i^+ := H_i \cap (\bigcup_{j=1}^{i-1} H_j)
\]

denote the intersection of the \( i \)th copy of \( H \) in \( T \) with the preceding \( i - 1 \) copies. For all \( 2 \leq i \leq r \) we construct the graph \( J_i \) from \( J_i^+ \) by removing the vertices of \( e_i \) if they are isolated, i.e.,

\[
J_i := (V(J_i^+) \setminus \{ v \in e_i \mid \deg_{J_i^+(v)} = 0 \}, E(J_i^+)) .
\]

Note that for all \( 2 \leq i \leq b \) the vertices of \( e_i \) must be isolated in \( J_i^+ \) since \( E(H_1), E(H_2), \ldots, E(H_b) \) pairwise only intersect in \( e_1 \). Hence, we have \( e_i \subseteq V(J_i^+) \) and \( e_i \cap V(J_i) = \emptyset \), which gives us

\[
\sum_{i=2}^{r} v(J_i^+) - v(J_i) \geq \sum_{i=2}^{b} v(J_i^+) - v(J_i) = \sum_{i=2}^{b} 2 = 2(b - 1) .
\] (4.4)

For every \( K \in K \) we define the parameter

\[
y(K) := |K \cap E(T)| ,
\]

which counts the number of edges in \( K \) that are part of some \( H_i \) as well. In the most intuitive case we have \( y(K) = 1 \) for all \( K \in K \). Nevertheless, some \( r \)-sets
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in $K$ may contribute more than just a single edge to $T$. Using $E(J_i) \cap \bar{K} = \emptyset$ for all $2 \leq i \leq r$ we obtain

$$r(e_H - 1) - \sum_{i=2}^{r} e(J_i) = \left| \bigcup_{i=1}^{r} E(H_i \setminus \{e_i\}) \right| = \sum_{K \in K \setminus \{\bar{K}\}} y(K) \tag{4.5}$$

We set

$$M := \sum_{K \in K \setminus \{\bar{K}\}} (y(K) - 1) = 1 + r(e_H - 1) - \sum_{i=2}^{r} e(J_i) - |K| \tag{4.5}$$

and obtain

$$|K| = 1 + \sum_{K \in K \setminus \{\bar{K}\}} (y(K) - y(K) + 1) = 1 + r(e_H - 1) - \sum_{i=2}^{r} e(J_i) - \sum_{K \in K \setminus \{\bar{K}\}} (y(K) - 1) = 1 + r(e_H - 1) - \sum_{i=2}^{r} e(J_i) - M \tag{4.6}$$

In this equation $M$ plays the role of a correction term. In the intuitive case where $y(K) = 1$ for all $K \in K$ we have $M = 0$.

Next we derive an upper bound for the number of vertices of $T^r$. These consist of the vertices of $T$ and further vertices from edges outside $T$. Since every such edge contributes at most $2$ vertices to $T^r$ the latter number is bound by

$$\sum_{K \in K \setminus \{\bar{K}\}} 2(r - y(K)).$$

This gives us

$$|V(K)| \leq |V(T)| + \sum_{K \in K} 2(r - y(K))$$

$$= rv_H - \sum_{i=2}^{r} v(J_i^+) + 2(r - y(K)) + 2 \sum_{K \in K \setminus \{\bar{K}\}} (r - 1 - y(K) - 1)$$

$$= rv_H - \sum_{i=2}^{r} v(J_i^+) + 2(r - y(K)) + 2(r - 1)(|K| - 1) - 2 \sum_{K \in K \setminus \{\bar{K}\}} (y(K) - 1)$$

$$= rv_H - \sum_{i=2}^{r} v(J_i^+) + 2(r - y(K)) - 2(r - 1) + 2(r - 1)|K| - 2M$$

$$\leq rv_H - \sum_{i=2}^{r} v(J_i^+) + \sum_{i=2}^{r} (v(J_i^+) - v(J_i) - 2(r - 1) + 2(r - 1)|K| - 2M$$

$$= -2(r - 1) + rv_H - \sum_{i=2}^{r} v(J_i) + 2(r - 1)|K| - 2M$$

$$= 2 + r(v_H - 2) - \sum_{i=2}^{r} v(J_i) + 2(r - 1)|K| - 2M \tag{4.7}$$
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We now show that for all $2 \leq i \leq r$ we have
\[ d(J_i) = \frac{e(J_i)}{v(J_i)} \leq \overline{m}_{2b}(F). \tag{4.8} \]

For this, we make a case distinction.

Case 1: $e(J_i)/v(J_i) \geq 1/2$. We have
\[ e(J_i) = \frac{r e(J_i)}{r v(J_i)} \overset{\text{Prop. 46}}{\leq} \frac{r e(J_i) - (r - 1)}{r v(J_i) - 2(r - 1)} = \frac{r e(J_i) - 1 + 1}{r v(J_i) - 2 + 2} \leq \frac{r e_H - 1 + 1}{r v_H - 2 + 2} = \overline{m}_{2b}(F), \]
where the second inequality follows from the fact that $J_i \subseteq F$.

Case 2: $e(J_i)/v(J_i) < 1/2$. Since $\overline{m}_{2b}(F) \geq 1/2$ by Corollary 48 we immediately get
\[ e(J_i)/v(J_i) \leq \overline{m}_{2b}(F). \]

This establishes (4.8).

We can now estimate the expectation of $X_T$, by using Lemma 46. To improve readability we set $p = m/n^2$ and obtain
\[
\mathbb{E}[X_T] \overset{\text{Lem. 46}}{=} O \left( n^{|V(K)| - 2(r-1)|K|}p^{|K|} \right) \overset{\text{Lem. 47}}{=} O \left( n^{2+r(e_H-2)-\sum_{i=2}^r e(J_i)-2M}p^{1+r(e_H-1)-\sum_{i=2}^r e(J_i)-M} \right) \overset{\text{Prop. 46}}{=} O \left( n^{2+r(e_H-2)}p^{1+r(e_H-1)}n^{-\sum_{i=2}^r e(J_i)}p^{-\sum_{i=2}^r e(J_i)}(n^2p)^{-M} \right) \overset{p \ll n^{-1/m_{2b}(F)}}{=} o \left( n^{-\sum_{i=2}^r e(J_i)}p^{-\sum_{i=2}^r e(J_i)}(n^2p)^{-M} \right) \overset{p \ll n^{-1/m_{2b}(F)}}{=} o \left( (n^2p)^{-M} \right). \]

W.l.o.g. we may assume $n^2p = \omega(1)$ since otherwise Painter will obviously survive the first $N$ steps a.a.s. with any strategy. This yields $\mathbb{E}[X_T] = o(1)$.

As indicated before Lemma 50 and 51 together imply that the $H$-avoidance strategy survives $N$ steps a.a.s. provided that $N \ll n^{2-1/m_{2b}(F)}$.

4.3 The Upper Bound

In this section we prove part (ii) of Theorem 40. For this, we assume the existence of a graph $F_- \subseteq F$ with $e_F - 1$ edges such that $m_2(F_-) \leq \overline{m}_{2b}(F)$. 

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Figure 4.3: A ‘threat’ $r \cdot F_-$ for $F = K_4$.

Throughout this section we fix such a graph $F_-$. We need to show that regardless of Painter’s strategy she will be forced to close a monochromatic copy of $F$ within any $N \gg n^{2-1/\pi_2(F)}$ steps a.a.s. In this proof we work with the graph process $(G_k)_{0 \leq k \leq N}$ which denotes the simple graphs underlying $(G'_{k})_{0 \leq k \leq N}$. Note that the probability distribution for $G_k$ is the same as for the random graph $G_{n, rk}$.

Our proof is built upon Theorem 32. We use a two-round approach and consider a relaxed game in which Painter is revealed the edges of the first $N/2$ steps at once, allowing her to color them offline. Afterwards the game continues normally, i.e., the remaining edges come in sets of size $r$ and Painter has to color them immediately using each of the $r$ colors exactly once in every step. Clearly, if Painter loses this modified game a.a.s. regardless of her strategy, she will also lose the balanced online edge coloring game.

Playing for $N$ steps results in a random graph with $m := rN$ edges. In the following we set $p = m/n^2$, thus having

$$p \gg n^{-1/\pi_2(F)} \geq n^{-\frac{r(r-2)+2}{r(r-1)^2}}. \quad (4.9)$$

Throughout the proof we only use the $G_{n,m}$ model, but sometimes use the parameter $p$ for a more natural notation. Furthermore, we write $r \cdot G$ to refer to the union of $r$ disjoint copies of some graph $G$.

We show that a.a.s. the first $N/2$ steps generate a lot of ‘threats’ for the remaining steps of the game. The threats we consider here are subgraphs isomorphic to $r \cdot F$ (see Figure 4.3). Note that if the $r$ copies of $F_-$ are of the same color and Painter is presented the $r$ edges that complete them to copies of $F$, we lose the game. We want to apply Theorem 32 to $r \cdot F_-$ and hence need to show that $m_2(r \cdot F_-) \leq m_2(F_-)$. By Lemma 49, there exists a subgraph that defines $m_2(r \cdot F_-)$ and is connected, thus also being a subgraph of $F_-$. This immediately yields $m_2(r \cdot F_-) = m_2(F_-)$.

Now Theorem 32 states that a.a.s. the random graph $G_{n,m/2}$, which represents $G_{n/2}$ (the situation after the first $N/2$ steps), contains $\Omega(n^{v(r \cdot F_-)}p^{e(r \cdot F_-)})$ monochromatic copies of $r \cdot F_-$. Clearly, there exists a color $s_0 \in \{1, 2, \ldots, r\}$ such that at least a $1/r$-fraction of these copies is of color $s_0$. 

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We now come to the analysis of the second round, i.e., the last $N/2$ steps. Assume the outcome of the first round to be fixed and let $M$ denote the number of copies of $r \cdot F_-$ in color $s_0$ at its end. We assume that we have

$$M = \Omega(n^{v(r \cdot F_-)} p^{e(r \cdot F_-)}) = \Omega(E[X_0; F_-]),$$

which is a.a.s. the case. We denote the copies of $r \cdot F_-$ by $[r \cdot F_-]_1, [r \cdot F_-]_2, \ldots, [r \cdot F_-]_M$. The edge sets that ‘complete’ these copies of $r \cdot F_-$ to copies of $r \cdot F$ are denoted by $T_1, T_2, \ldots, T_M \subseteq E(K_n)$, i.e., for all $1 \leq i \leq M$ we have $|T_i| = r$ and

$$(V([r \cdot F_-]_i), E([r \cdot F_-]_i) \cup T_i) \cong r \cdot F.$$  

Note that depending on $F$ the $T_i$’s need not be unique (see Figure 4.4 for an example). In such a case we fix one of the possible sets arbitrarily. Clearly, if Painter is presented one of the sets $T_1, T_2, \ldots, T_M$ in one of the steps $N/2 + 1, \ldots, N$ in the second round, she is forced to close a copy of $F$ in color $s_0$ and therefore loses the game. We prove that this is a.a.s. the case.

For each set $T_i$ we introduce an indicator variable $Z_i$ for the event that $T_i$ appears as edge set $E_k$ in some step of the second round. Note that there might exist $i \neq j$ with $T_i = T_j$ and thus also $Z_i \equiv Z_j$. We set $Z := \sum_{i=1}^M Z_i$ and show that a.a.s. $Z > 0$ holds. We do this with the second moment method. In order to calculate $E[Z]$ we inspect $P[Z_i = 1]$.

We may assume that $m = o(n^2)$ since otherwise we can for large enough $n$ a.a.s. guarantee the appearance of an arbitrary fixed graph in $G_{n,m}$, in particular one that is not colorable with $r$ colors without creating a monochromatic copy of $F$. Therefore, we still have $\Theta(n^2)$ edges in each of the $N$ steps that did not appear in the random graph process yet. Since $Z_i = 1$ if and only if Painter is presented $T_i$ in one of the remaining $N/2 = \Theta(m)$ steps, we obtain

$$P[Z_i = 1] = \Theta \left( m \cdot \frac{1}{(n^2)} \right) = \Theta(mn^{-2r}).$$  

(4.11)
This yields

\[ E[Z] = \sum_{i=1}^{M} P[Z_i = 1] = M \Theta(mn^{-2r}) \]

\[(4.10)\]

\[ \Omega(E[X_{r,F_i} \cdot mn^{-2r}) \]

\[ = \Omega(n^{r(r-F_i)} \cdot p^{-1} \cdot mn^{-2r}) \]

\[ \Omega(n^{r-1} \cdot mn^{-2r}) \]

\[(4.12)\]

\[ \equiv \omega(n^{2r-2} \cdot p^{-1} \cdot mn^{-2r}) \]

\[(4.13)\]

\[ = \omega(1) \]

It remains to show that \( \text{Var}[Z] = o(E[Z]^2) \). Note that for \( i,j \) with \( T_i \cap T_j = \emptyset \) the random variables \( Z_i \) and \( Z_j \) are negatively correlated, i.e.,

\[ E[Z_i Z_j] - E[Z_i] E[Z_j] \leq 0 \]

\[(4.13)\]

We have

\[ \text{Var}[Z] = E[Z^2] - E[Z]^2 \]

\[ = \sum_{i,j=1}^{M} (E[Z_i Z_j] - E[Z_i] E[Z_j]) \]

\[(4.13)\]

\[ \leq \sum_{i,j=1}^{M} (E[Z_i Z_j] - E[Z_i] E[Z_j]) \]

\[ \leq \sum_{i,j=1}^{M} E[Z_i Z_j] \]

\[ \sum_{i,j=1}^{M} E[Z_i Z_j] \]

where the last equation follows from the observation that, if for a pair \( i,j \) we have that \( T_i \) and \( T_j \) share some but not all edges (i.e. \( T_i \cap T_j \neq \emptyset \) and \( T_i \neq T_j \)), then \( E[Z_i Z_j] = 0 \) since it is impossible for both \( T_i \) and \( T_j \) to appear as the set of edges in some of the steps in the game.

We are left to deal with pairs \( i,j \) for which \([r \cdot F_i]\) and \([r \cdot F_j]\) produce the same threat (i.e. \( T_i = T_j \)), which immediately implies \( Z_i \equiv Z_j \) and hence \( E[Z_i Z_j] = E[Z] \). Note that \([r \cdot F_i] \cap [r \cdot F_j] \neq \emptyset \), since they do overlap in the 2r endvertices of the edges of \( T_i = T_j \). Of course, they may also overlap in further vertices and edges. We set \( J_{i,j} = [r \cdot F_i] \cap [r \cdot F_j] \) to be the intersection graph (see Figure 4.5) and split the above sum with respect to the type of intersection.
between \([r \cdot F_-]_i\) and \([r \cdot F_-]_j\):

\[
\text{Var}[Z] \leq \sum_{i,j=1}^{M} \mathbb{E}[Z_i Z_j] \\
= \sum_{J \subseteq r \cdot F_-} \sum_{i,j=1}^{M} \mathbb{E}[Z_i] \\
\leq \sum_{J \subseteq r \cdot F_-} \sum_{i,j=1}^{M} \Theta(mn^{-2r}) \quad (4.11)
\]

For \(J \subseteq r \cdot F_-\) let \(M_J\) denote the number of pairs \(i,j\) for which \(T_i = T_j\) and \(J_{i,j} \cong J\). In other words, \(M_J\) counts the number of \(s_0\)-colored subgraphs in \(G_{N/2}\) consisting of two copies of \(r \cdot F_-\) which form identical threats and overlap in a copy of \(J\). Since the \(M_J\)'s only depend on the outcome of the first round, they are fixed. We show that for all \(J \subseteq r \cdot F_-\) we a.a.s. have

\[
M_J = o(\mathbb{E}[X_{r \cdot F_-}]^2 \cdot mn^{-2r}) \quad (4.15)
\]

Note that this involves an inspection of the first round again. Let \(J \subseteq r \cdot F_-\) be fixed. Since we are only interested in pairs \(i,j\) with \(T_i = T_j\) we may restrict ourselves to subgraphs \(J\) that contain the endvertices of the \(2r\) edges that complete \(r \cdot F_-\) to \(r \cdot F\). Formally, setting \(T = E(r \cdot F) \setminus E(r \cdot F_-)\) we assume

\[
V(T) \subseteq V(J) .
\]

This allows us to add the edges of \(T\) to \(J\) resulting in a graph \(J^+\). Note that we have \(v(J^+) = v(J)\) and \(e(J^+) = e(J) + r\). Moreover, the graph \(J^+\) can be split into \(r\) components \(J_1^+, J_2^+, \ldots, J_r^+\) such that \(J_i^+ \subseteq F\) for all \(1 \leq i \leq r\) which
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gives us
$$\frac{r(e(J_i^+)) - 1}{r(v(J_i^+)) - 2} + 2 \leq m_{2b}^r(F)$$

and hence also
$$p^{-1} < n^{\frac{r(e(J_i^+)) - 2 + 2}{r(v(J_i^+)) - 2 + 2/r}} = n^{\frac{v(J_i^+)}{r(J_i^+)} - 1 + 1/r}.$$  (4.16)

We obtain
$$E[M_J] = \Theta(n^{2v(rF_r) - v_J} p^{2e(rF_r) - e_J})$$
$$= \Theta(E[X_{r,F_1}]^2 \cdot n^{-v_J} p^{-e_J})$$
$$= \Theta(E[X_{r,F_1}]^2 \cdot n^{-v(J^+)} p^{-(J^+)-r})$$
$$= \Theta(E[X_{r,F_1}]^2 \cdot n^{-\sum_{i=1}^r v(J_i^+)} p^{-\sum_{i=1}^r [v(J_i^+)] - 1} / p^r)$$
$$\leq \Theta(E[X_{r,F_1}]^2 \cdot \prod_{i=1}^r \left(n^{-v(J_i^+)} + [v(J_i^+)] - 2 + 2/r\right) \cdot p$$
$$= \Theta(E[X_{r,F_1}]^2 \cdot (n^{-2+2/r})^r p$$
$$m^{ag} \leq n^{2r} E[X_{r,F_1}]^2 \cdot mn^{-2r}.$$  (4.15)

By the methods of first and second moment this proves that (4.15) a.a.s. holds.

Since the number of subgraphs $J \subseteq rF_r$ only depends on $F$ and $r$ and is thus constant, this bound holds a.a.s. for all subgraphs simultaneously.

This yields with (4.14) that a.a.s.
$$\text{Var}[Z] \leq \sum_{J \subseteq rF_r} \sum_{i,j=1}^M \Theta(mn^{-2r})$$
$$= \sum_{J \subseteq rF_r \atop V(T) \subseteq V(J)} M_J \Theta(mn^{-2r})$$
$$= o(E[X_{r,F_1}]^2 \cdot (mn^{-2r})^2) \leq o(E[Z]^2).$$

The second moment method then shows that a.a.s. we have $Z > 0$, i.e., Painter is a.a.s. given at least one of the edge sets $T_1, T_2, \ldots, T_M$ in one of the steps $N/2 + 1, \ldots, N$ of the second round, which forces her to lose the game.  \[\square\]

4.4 Application to Cycles and Cliques

Having proved Theorem 30 we can now show threshold phenomena for the duration of $\text{OEC}_b(F, r, n)$ by verifying the condition from the upper bound.
To obtain concrete bounds for some important graphs, we present proofs for Corollary 41 and 42.

**Proof of Corollary 41** Let \( t \geq 3 \) and \( r \geq 1 \) be integers. We need to show the existence of a graph \( F_- \subseteq C_t \) with \( e(C_t) - 1 \) edges that satisfies \( m_2(F_-) \leq \overline{m}_{2b}(C_t) \). We have
\[
m_2(F_-) = \frac{t-2}{t-2} = 1 \leq \frac{r(t-1)+1}{r(t-2)+2} = \overline{m}_{2b}(C_t) .
\]

**Proof of Corollary 42** Let \( t \geq 3 \) and \( r \geq t \) be integers. We need to show the existence of a graph \( F_- \subseteq K_t \) with \( e(K_t) - 1 \) edges that satisfies \( m_2(F_-) \leq \overline{m}_{2b}(K_t) \). Clearly, we have
\[
r \geq t - \frac{2}{t-2} = \frac{t(t-2) - 2}{t-2} = \frac{2\left(\binom{t}{2} - 2\right) - (t-2)}{t-2} = \frac{2\left(\binom{t}{2} - 2\right)}{t-2} - 1,
\]
which implies
\[
\frac{r+1}{2} \geq \frac{\binom{t}{2} - 2}{t-2} .
\]
Using this together with Proposition 47 we obtain
\[
m_2(F_-) = \frac{\binom{t}{2} - 2}{t-2} = \frac{r\left(\binom{t}{2} - 1\right) - r}{r(t-2)} \overset{\text{Prop. 47}}{\leq} \frac{r\left(\binom{t}{2} - 1\right) + 1}{r(t-2) + 2} = \overline{m}_{2b}(K_t) .
\]
Chapter 5

A General Upper Bound on the Duration of $\text{OEC}_b(F, r, n)$

In this chapter we generalize the argument from the upper bound proof as presented in the previous chapter to obtain a non-trivial upper bound for arbitrary graphs and an arbitrary number of colors. Again, we let $F$ be a graph and $r \geq 1$ and integer both of which are fixed throughout this chapter.

We point out that the reason for the proof not to be applicable to all graphs is the condition of Theorem 32, that is, the existence of a graph $F_\sim \subseteq F$ with $e_F - 1$ edges such that $m_2(F_\sim) \leq \pi_{2b}(F)$. The basic idea of this chapter’s proof is to not only remove a single, but sufficiently many, say $s$, edges from $F$ until this condition is satisfied. The second part of the proof then needs $s$ rounds instead of one since we not only need to prove that Painter will close the last edge of $F$, but the remaining $s$ edges.

5.1 A General Upper Bound

We start over with some required definitions and notation.
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Figure 5.1: Gluing the graphs $F \setminus \{e_1\}, F \setminus \{e_1, e_2\}, \ldots, F \setminus \tilde{S}$ together at the corresponding edges

**Gluing graphs together**

The following definition is illustrated in Figure 5.1. Let $\tilde{S} = (e_1, e_2, \ldots, e_s) \subseteq E(F)$ (where $s \in \mathbb{N}$) be a sequence of $s$ distinct edges. We define the graph

$$F_{\tilde{S}} := \begin{cases} 
(e \cup \bigcup_{i=1}^{r} (V_{1,i} \cup V_i), \{e\} \cup \bigcup_{i=1}^{r} E_i) & \text{if } s > 1 \\
(e \cup \bigcup_{i=1}^{r} V_{1,i}, \{e\} \cup \bigcup_{i=1}^{r} E_i) & \text{if } s = 1
\end{cases},$$

where for all $1 \leq i \leq r$ we have

$$(e \cup V_{1,i}, E_i \cap (e \cup V_{1,i}) \bigcap \frac{V_{1,i}}{2}) \cong F \setminus \{e_1\} \quad \text{and} \quad (V_{1,i} \cup V_i, E_i) \cong F_{(e_2, e_3, \ldots, e_s)}.$$

Note that we have

$$e(F_{\tilde{S}}) = r^s(e(F) - s) + 1 + r + \ldots + r^{s-1} = r^s(e(F) - s) + \frac{r^s - 1}{r - 1}, \quad (5.1)$$

and

$$v(F_{\tilde{S}}) = r^s(v(F) - 2) + 2. \quad (5.2)$$

**A General Upper Bound**

**Definition 5.2.** We set

$$\overline{m}_{\tilde{S}}(F) := \min_{\tilde{S} \subseteq E(F), m_{\tilde{S}}(F \setminus \tilde{S}) \leq m(F_{\tilde{S}})} m(F_{\tilde{S}}).$$
Chapter 5. A General Upper Bound on the Duration of \( OEC_b(F, r, n) \)

**Theorem 5.3.** Painter will a.a.s. lose \( OEC_b(F, r, n) \) within any \( N \gg n^{2-1/\overline{m}_b^*(F)} \) steps regardless of her strategy.

*Proof.* Let \( N \gg n^{2-1/\overline{m}_b^*(F)} \). We need to show that regardless of Painter’s strategy she will be forced to create a monochromatic copy of \( F \) within \( N \) steps of \( OEC_b(F, r, n) \). We will basically use the same technical machinery as in the previous chapter, but in a more complicated framework. Hence, many of the calculations state analogous results and might look familiar to the reader. As in the previous chapter we set \( m := rN \) and \( p = m/n^2 \), and denote the graph process of the game by \( (G_k^r)_{0 \leq k \leq N} \) or \( (G_k)_{0 \leq k \leq N} \) depending on whether we refer to \( r \)-matched or simple graphs. The edge-sets that are presented to Painter in the process are denoted by \( E_1, E_2, \ldots, E_N \). Let

\[
\bar{S} = (e_1, e_2, \ldots, e_s) := \arg\min_{S \subseteq E(F), m_2(F \setminus S) \leq m(F_p)} m(F_S).
\]

Furthermore, we define for all \( 1 \leq i \leq s \) the graph \( F_{-i} := F \setminus \{e_1, e_2, \ldots, e_i\} \).

We split the \( N \) steps into \( s + 1 \) equally large rounds by defining for all \( 0 \leq i \leq s \) the parameter

\[
N(i) := (s - (i - 1)) \cdot \frac{N}{s + 1}.
\]

Speaking intuitively, we prove by induction that for all \( 0 \leq i \leq s \) we are a.a.s. \( i \) edges away from a monochromatic copy of \( F \) after \( N(i) \) rounds, which implies Theorem 5.3 for \( i = 0 \).

We start the induction with the case \( i = s \). As in the previous proof we use Theorem 32 for this first round (steps \( N(s + 1) \) to \( N(s) \)) to ensure that we have a lot of ‘threats’ for the remaining rounds. More formally, we consider ‘threats’ of the form \( F_i^* := r^i \cdot F_{-s} \). The intuition behind this graph is that in every round an \( 1/r \)-fraction of the copies of \( F_{-s} \) might become one edge closer to a copy of \( F \), if Painter is presented appropriate edge-sets, such that after \( s \) rounds exactly 1 such copy might have developed into \( F \) (see Figure 5.2). Note that the definition of \( \overline{m}_b^*(F) \) ensures that \( m_2(F_{-s}) \leq \overline{m}_b^*(F) \).

Since \( m_b(F_{-s}) = m_b(F_{-s}) \) by Lemma 49, Theorem 32 ensures that a.a.s. there exists a color \( s_0 \in \{1, \ldots, r\} \) such that we have \( \Omega(n^{2-m_b^*(F_p)}) = \Omega(E[X_{F_s}]) \) \( s_0 \)-colored copies of \( F_{-s} \) in \( G_{N(s)} \).

In the induction steps we now prove lower bounds on the number of such threats that develop to ‘more dangerous’ graphs. More, formally we define for all \( 0 \leq i \leq s - 1 \) the graph

\[
F_i^* := r^i \cdot F_{-s} \cup \bigcup_{j=i+1}^{s} (r^j - r^{j-1}) \cdot F_{-s}.
\]

which denotes a threat that has already developed up to level \( i \), i.e., there are just \( i \) edges missing to a copy of \( F \). For example, only considering the black
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Figure 5.2: A monochromatic graph of the form $r^s \cdot F_{-s}$ (black edges) for $F = K_4, s = 2, r = 3$ and an example for edge-sets that force Painter to create a monochromatic copy of $F$ (gray edges).

edges in Figure 5.2 yields a graph $F^*_2$ (for $F = K_4$). Including the 3 rightmost dashed gray edges gives a graph $F^*_1$, and further adding the solid gray edge in the bottom right corner makes a graph $F^*_0$.

We call $T \subseteq (\binom{(i-r+1)i}{i})$ (where $1 \leq i \leq s$) a level-$i$-threat for a monochromatic copy of $F^*_i$ if

- $|T| = r^{i-1}$, and
- $(V(F^*_i), E(F^*_i) \cup E(T)) \cong r^i \cdot F_{-(i-1)} \bigcup_{j=i+1}^{s} (r^{i-1} - r^{j-1}) \cdot F_{-j}$,

where $E(T) = \bigcup_{A \in T} A$. For example, in Figure 5.2 the dashed gray edges form a level-2-threat. Note that if Painter is presented all $r^{i-1}$ $r$-sets of a level-$i$-threat, she is forced to create a monochromatic copy of $F^*_{i-1}$. If she is presented a level-1-threat she is forced to create a monochromatic copy of $F$.

The formal setup for the induction steps is as follows: We assume the outcome of the first round (steps $N(s+1)$ to $N(s)$) to be fixed and denote by $M(s)$ the number of $s_0$-colored copies of $F^*_s$ in $G_{N(s)}$. Moreover, we assume $M(s) = \Omega(\mathbb{E}[X_{F^*_s}])$, which is a.a.s. the case as mentioned before. We fix an arbitrary labeling $F^*_{s1}, F^*_{s2}, \ldots, F^*_{sM(s)}$ for the $M(s)$ copies of $F^*_s$. For all $0 \leq i < s$ we define $M(i)$ inductively as follows:

Assume that $F^*_i, F^*_{i+1,1}, \ldots, F^*_{i+1,M(i+1)}$ are $s_0$-colored copies of $F^*_{i+1}$. For each such graph $F^*_{i+1,j}$ we fix an arbitrary level-$(i+1)$-threat $T_{i+1,j}$. By $M(i)$ we denote the random variable that counts the number of these threats from
which each of the \( r^i \) \( r \)-sets of edges is presented Painter between step \( N(i+1) \) and \( N(i) \). This results in \( M(i) \) \( s_0 \)-colored copies of \( F_i \), which we denote by \( F_{i,1}^*, F_{i,2}^*, \ldots, F_{i,M(i)}^* \). We prove by induction that for all \( 0 \leq i < s \) we have a.a.s.

\[
M(i) = \Omega \left( \mathbb{E}[X_{(i)}][n^{2-2r}p \sum_{j=i}^{i+1} r^j] \right).
\]

Note that this yields a.a.s.

\[
M(0) = \Omega \left( \mathbb{E}[X_{(0)}][n^{2-2r}p \sum_{j=0}^{0} r^j] \right)
= \Omega \left( n^{r^*(F)} p^{r^*(e(F))} n^{-2r^*+2} p^{r^*/r} \right)
= \Omega \left( n^{r^*(e(F)-2)} p^{r^*(e(F))} n^{-r^*/r} \right)
\]

which implies that at least one \( s_0 \)-colored copy of \( F \) was created within \( N(0) = N \) steps.

It remains to show (5.3). Let \( 0 \leq i < s \) and assume that

\[
M(i+1) = \Omega \left( \mathbb{E}[X_{(i+1)}][n^{2-2r}p \sum_{j=i+1}^{i+1} r^j] \right),
\]

which is a.a.s. the case. In order to use the second moment method we introduce indicator variables \( Z_j \) (where \( 1 \leq j \leq M(i+1) \)) for the event that all \( r^i \) edge-ssets in \( T_{i+1,j} \) are presented Painter between step \( N(i+1) \) and \( N(i) \). Note that there might be \( j \neq k \) with \( T_{i+1,j} = T_{i+1,k} \) and thus also \( Z_j \equiv Z_k \). By definition we have \( M(i) = \sum_{j=1}^{M(i+1)} Z_j \) and

\[
\mathbb{E}[M(i)] = \sum_{j=1}^{M(i+1)} \prod_{T \in T_{i+1,j}} \sum_{k=N(i)+1}^{N(i+1)} \mathbb{P}[E_k = T]
= \sum_{j=1}^{M(i+1)} \prod_{T \in T_{i+1,j}} \sum_{k=N(i)+1}^{N(i+1)} \Theta \left( \frac{1}{n^2} \right)
= \sum_{j=1}^{M(i+1)} \prod_{T \in T_{i+1,j}} (N(i) - N(i+1)) \Theta \left( \frac{1}{n^2} \right)
= \sum_{j=1}^{M(i+1)} \Theta \left( \left( \frac{N}{n^2} \right)^{r^j} \right)
= M(i+1) \cdot \Theta \left( \left( \frac{n^2 p}{n^2} \right)^{r^{i+1}} \right)
= \Omega \left( \mathbb{E}[X_{(i)}][n^{2-2r}p \sum_{j=i+1}^{i+1} r^j] \right).
\]
It remains to show $\text{Var}[M(i)] = o(\mathbb{E}[M(i)]^2)$. We have

$$\text{Var}[M(i)] = \mathbb{E}[M(i)^2] - \mathbb{E}[M(i)]^2$$

$$= \sum_{j,k=1}^{M(i+1)} \mathbb{E}[Z_j Z_k] - \mathbb{E}[Z_j] \mathbb{E}[Z_k]$$

$$\leq \sum_{j,k=1}^{M(i+1)} \mathbb{E}[Z_j Z_k]$$

$$\leq \sum_{j,k=1}^{M(i+1)} \mathbb{E}[Z_j Z_k]$$

$$= \sum_{j,k=1}^{M(i+1)} \sum_{T_i \in \mathcal{F}_j} \sum_{T_i \in \mathcal{F}_k} M(J,T_i) \Theta \left((n^{2-2r} p)^{2r^j - |T_i|}\right)$$

where $M(J,T_i)$ denotes the number of pairs $j,k$ for which $F_{i+1,j} \cap F_{i+1,k} \cong J$ and $T_{i+1,j} \cap T_{i+1,k} \cong T_i$. It remains to show that for all $(J,T_i) \subseteq (F_{i+1},(V(F_{i+1}))_N)$, we have a.a.s.

$$M(J,T_i) = o \left( \mathbb{E}[X_{F_{i+1}}(n^{2-2r} p)^{\sum_{j,k=1}^{M(i+1)} 2r^j - |T_i|}\right) \right).$$

Since the number of such pairs $(J,T_i) \subseteq (F_{i+1},(V(F_{i+1}))_N)$ only depends on $F$ and $r$ and is thus constant, this will ensure that this bound a.a.s. holds for all pairs simultaneously.

Let $(J,T_i) \subseteq (F_{i+1},(V(F_{i+1}))_N)$. We partition the edges of $J$ into $E(J_0) = E(J) \cap E(F_{i+1})$ and $E(J_1) = E(J) \setminus E(J_0)$. Speaking intuitively, the edges $E(J_0)$ were already present in $G_{N(s)}$ (in Figure 5.2, these are the gray edges) and the edges $E(J_1)$ are those which made a copy of $F_{i+1}$ to a more dangerous graph and which came into the graph in later rounds, that is between step $N(s)$ and $N(i)$ (in Figure 5.2, these are some of the black edges). We introduce $F_J$ to denote the graph that consists of two copies of $F_{i+1}$ that overlap in a copy of $J$. Note that for a pair $j,k$ with $F_{i+1,j} \cap F_{i+1,k} \cong J$ and $T_{i+1,j} \cap T_{i+1,k} \cong T_i$ the graph $F_{i+1,j} \cup F_{i+1,k}$ corresponds to a $s_0$-colored copy of $F_J$ in $G_{N(i+1)}$. Since each $s_0$-colored copy of $F_J$ may only generate a constant number of pairs $j,k$ with the desired properties (depending on the size of automorphism class of $F$ and the number of possibilities to assign the subgraphs of $F$ contained in $F_J$ to the one or the other copy of $F_{i+1}$), it suffices to bound the number of $s_0$-colored copies of $F_J$ in $G_{N(i+1)}$. We look at the $r$-matched graph induced by a $s_0$-colored copy of $F_J$ in $G_{N(i+1)}$, that is, $F_{J}^{s_0} := (V(K(E_{F_J})),K(E_{F_J}))$. We calculate $|K(E_{F_J})|$. Since every edge in $F_J$ is assigned the color $s_0$ we have $K(e) \neq K(e')$ for all
Clearly, we have \( |K(E(F_J))| = 2r^s(e_F - s) - e(J_0) + 2 \sum_{l=i+1}^{s-1} r^l - e(J_1) \). \( \text{(5.5)} \)

We also need to bound the number of vertices of \( F_J \). These consist of the vertices of \( F_J \) and the vertices that belong to the edges \( K(e) \setminus \{e\} \) for all \( e \in E(F_J) \). We point out that every set \( K(e) \) may add at most \( 2(r-1) \) further vertices to \( F_J \).

Note that for the \( 2 \sum_{l=i+1}^{s-1} r^l - e(J_1) \) edges in \( E(F_J) \) which, by intuition, appear in the graph process after step \( N(s) \) the set \( K(e) \) belongs to a level-\( l \) threat for some \( l > i + 1 \). Hence, for such an edge \( K(e) \) is located within the vertices of \( F_J \) and thus does not contribute further vertices to \( F_J \). Altogether, we obtain

\[
|V(F_J)| \leq 2r^s v_F - v(J) + 2(r-1)(2r^s(e_F - s) - e(J_0)) \quad \text{(5.6)}
\]

Using Lemma 46 we obtain that we have in expectation at most

\[
\mathcal{O}(n^{|V(F_J)| - |K(E(F_J))|} 2r^{|K(E(F_J))|})
\]

\[
= \mathcal{O}
\left(n^{|V(F_J)| - 2(r-1)|K(E(F_J))|} p^{|K(E(F_J))|}\right)
\]

\[
= \mathcal{O}(n^{2r^s v_F - v(J) - 2(r-1)(2 \sum_{l=i+1}^{s-1} r^l - e(J_1))})
\]

\[
= \mathcal{O}
\left(\left|\mathbb{E}[X_{F_J}]^2(n^{-2(r-1)p}\sum_{l=i+1}^{s-1} 2r^l + |T'|)\right|\right)
\]

\[
\left(n^{-v(J)p} n^{-2(r-1)p} e(J_1) + |T'|\right) = \omega(1) \quad \text{(5.7)}
\]

since this bounds the expected number of copies of \( F_J \) in \( G_{N(i+1)} \). It remains to show

\[
o \left(\left|\mathbb{E}[X_{F_J}]^2(n^{-2(r-1)p}\sum_{l=i+1}^{s-1} 2r^l + |T'|)\right|\right)
\]

which together with the methods of first and second moment implies 5.4.

Clearly, we have \( J \subseteq F_J^* \). We now describe how \( F_J^* \) can be modified in a natural way such that the modified graph \( \tilde{F}_J^* \) is a subgraph of \( F_J^* \). The result of this process is illustrated in Figure 5.3 for the graph of Figure 5.2. Let \( \tilde{T}_s, \tilde{T}_{s-1}, \ldots, \tilde{T}_1 \subseteq (\mathbb{C}^\mathbb{Z}) \) such that for all \( 1 \leq l \leq s \) we have that \( T_l \) is a level-\( l \) threat for the copy of \( F_J^* \) contained in \( F_J^* \). We now generate \( \tilde{F}_J^* \) from \( F_J^* \) by merging the vertices along these threats, i.e., for all \( 1 \leq l \leq s \) and \( T \in \tilde{T}_l \) we choose \( V_1, V_2 \subseteq V(T) \) arbitrarily such that \( V_1 \cup V_2 = T \) and \( |V_1 \cap e| = |V_2 \cap e| = 1 \) for all \( e \in T \), and then merge all the vertices in \( V_1 \) together into a single vertex, as well as we do for the vertices in \( V_2 \). Note that after this merging process \( \tilde{F}_J^* \) is a subgraph of \( F_J^* \), and that we have \( e(\tilde{F}_J^*) = e(F_J^*) \) and \( v(\tilde{F}_J^*) = v(F_J^*) - \frac{r^s - 1}{r^s - 1} 2(r-1) \).
Chapter 5. A General Upper Bound on the Duration of OECₜ(F, r, n)

Figure 5.3: Merging the vertices along the r-sets of edges of the threats

We point out that we can apply the same procedure to J as far as possible, i.e., as far as the corresponding vertices are in the copies of $F_{-s}, F_{-(s-1)}, \ldots, F_{-1}$ or $F$ respectively. We call the resulting graph $\tilde{J}$. Remember that we are only interested in graph $J$ that represent the intersection between two copies of $F_{i+1,j}$ and $F_{i+1,k}$ for some $1 \leq j < k \leq M(i+1)$ (for all other graphs $J$ we have $M(J, T \cap) = 0$). Note that this implies that for all edges $e \in e(J)$ the corresponding r-set $K(e)$ of edges is located within the vertices of $J$, i.e., $V(K(e)) \subseteq V(J)$ since $K(e)$ must be part of some previous threat of $F_{i+1,j}$ and $F_{i+1,k}$. For the same reason we have $V(T) \subseteq V(J)$ for all $T \in T^\cap$. This ensures that we can reduce the number of vertices by at least $(e(J) + |T^\cap|)|2(r-1)$ in this merging process. Moreover for every $T \in T^\cap$ we may add one further edge to $\tilde{J}$ between the two vertices that result from its merging. Altogether, we obtain

\[ v(\tilde{J}) \leq v(J) - (e(J) + |T^\cap|)2(r-1), \]

and

\[ e(\tilde{J}) = e(J) + |T^\cap|. \]

Since $\tilde{J} \subseteq F_S$ we obtain

\[ n^{e(J)}p^{e(J)}(n^{-2(r-1)}p)^{e(J) + |T^\cap|} \geq n^{e(J) - 2(r-1)(e(J) + |T^\cap|)}p^{e(J) + e(J) + |T^\cap|} \]

\[ = n^{e(\tilde{J})}p^{e(\tilde{J})} \]

\[ p \geq n^{-1/\delta_1(r)} \gg n^{-1/\delta(\tilde{J})} \]

This is establishes (5.7) and therefore finishes the proof. \qed
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5.2 Cliques of Arbitrary Size with an Arbitrary Number of Colors

In this section we apply Theorem 53 to obtain concrete upper bounds for cliques of arbitrary size and an arbitrary number of colors. In fact, for these special cases the bounds were already proved by Krivelevich, Sudakov and Loh in [10].

Corollary 54. Let $r \geq 1$ and $t \geq 4$. Then Painter will a.a.s. close a monochromatic copy of $K_t$ within any $N \gg n^{2-1/\theta}$ steps of OEC$_b(K_t, r, n)$, where

$$\theta(t, r) = \frac{r^s\left(\binom{t}{2} - s\right) + r^{s-1}}{r^s(t - 2) + 2} \quad \text{for} \quad s(t, r) = \lfloor \log_r((r - 1)t + 1) \rfloor.$$ 

In the remainder of this section we fix $r \geq 1$ and $t \geq 4$ and set $F := K_t$ and $s := s(t, r)$. To prove this corollary we first need to state 3 technical lemmas which provide us with some graph properties of $K_t$ that we will need in our proof.

Lemma 55. For all $\vec{S} \subseteq E(F)$ with $|\vec{S}| \leq s$ the graph $F \setminus \vec{S}$ is 2-balanced.

Proof. We need to show that for all $D \subseteq F \setminus \vec{S}$ we have $d_2(D) = \frac{e(D) - 1}{v(D) - 2} \leq \frac{e(F \setminus \vec{S}) - 1}{v(F \setminus \vec{S}) - 2} = d_2(F \setminus \vec{S})$. Let $D \subseteq F \setminus \vec{S}$. W.l.o.g. we may assume $e(D)/v(D) \geq 1/2$. It suffices to show

$$\frac{e(F \setminus \vec{S}) - e(D)}{v(F \setminus \vec{S}) - v(D)} \geq \frac{e(D) - 1}{v(D) - 2} \quad (5.8)$$

since this implies

$$d_2(D) = \frac{e(D) - 1}{v(D) - 2} \leq \frac{e(F \setminus \vec{S}) - 1 + e(F \setminus \vec{S}(i, j)) - e(D)}{v(D) - 2 + v(F \setminus \vec{S}) - v(D)} = d_2(F \setminus \vec{S}).$$

Note that we have

$$\frac{e(F \setminus \vec{S}) - e(D)}{v(F \setminus \vec{S}) - v(D)} \geq \frac{\binom{t}{2} - s - \frac{v(D)}{2}}{t - v(D)}, \quad (5.9)$$

and

$$\frac{e(D) - 1}{v(D) - 2} \leq \frac{\frac{v(D)}{2} - 1}{v(D) - 2}. \quad (5.10)$$
Moreover,

\[
s \leq \lfloor \log_r((r-1)t+1) \rfloor
\]

If \( t \geq 4 \), then

\[
s \leq \lfloor \log_r((r-1)t + \frac{t}{4}) \rfloor
\]

Therefore

\[
\leq \lfloor \log_r((r-3/4) + \log_r t) \rfloor
\]

\[
\leq \lfloor \log_r t \rfloor
\]

\[
\leq t - 1
\]

\[
= \frac{t(t-1)}{2} - \frac{(t-1)(t-2)}{2}
\]

\[
\leq \left( \frac{t}{2} \right) - \left( \frac{v(D)}{2} \right)
\]

\[
\leq \left( \frac{t}{2} \right) - \frac{t-2}{v(D)-2} \left( \frac{v(D)}{2} \right) + \frac{t-v(D)}{v(D)-2}
\]

\[
= \left( \frac{t}{2} \right)(v(D)-2) + (t-v(D)) - (t-2)\left( \frac{v(D)}{2} \right)
\]

\[
= \frac{\left( \left( t \right) - \left( \frac{v(D)}{2} \right) \right)(v(D)-2) - (\left( \frac{v(D)}{2} \right) - 1)(t-v(D))}{v(D)-2}
\]

Rearranging the terms gives us

\[
\frac{\left( t \right) - s - \left( \frac{v(D)}{2} \right)}{t-v(D)} \geq \frac{\left( \frac{v(D)}{2} \right) - 1}{v(D)-2}
\]

which together with (5.9) and (5.10) immediately implies (5.8).

**Lemma 56.** For all \( \vec{S} \subseteq E(F) \) with \(|\vec{S}| < s\) we have \( m_2(F \setminus \vec{S}) \geq d(F_{\vec{S}}) \), where we can only have equality if \(|\vec{S}| = s-1\). Moreover, if \(|\vec{S}| = s\) we have \( m_2(F \setminus \vec{S}) < d(F_{\vec{S}}) \).

**Proof.** Let us first assume that \( \tilde{s} := |\vec{S}| < s \). We want to show \( m_2(F \setminus \vec{S}) \geq d(F_{\vec{S}}) \). Note that

\[
d(F_{\vec{S}}) = \frac{r^\tilde{s}(\left( \frac{t}{2} \right) - \tilde{s}) + 1 + \ldots + r^{\tilde{s}-1}}{r^{t-2} + 2} = \frac{r^\tilde{s}(\left( \frac{t}{2} \right) - \tilde{s}) + r^{\tilde{s}-1}}{r^{t-2} + 2}
\]

By Lemma 55 we also have

\[
m_2(F \setminus \vec{S}) = \frac{\left( \frac{t}{2} \right) - \tilde{s} - 1}{t-2}
\]

hence we need to show

\[
(t-2) \left[ r^\tilde{s} \left( \frac{t}{2} - \tilde{s} \right) + \frac{r^{\tilde{s}-1}}{r-1} \right] \leq \left[ \left( \frac{t}{2} \right) - \tilde{s} - 1 \right] (r^\tilde{s}(t-2) + 2). \tag{5.11}
\]

We point out that \( \tilde{s} < s \) implies

\[
\tilde{s} \leq \frac{t-2}{2}, \tag{5.12}
\]
which can be seen by
\[ \tilde{s} \leq \log_r((r - 1)t + 1) - 1 \leq \lfloor \log_r t \rfloor \leq \frac{t - 2}{2} \]
for \( r \geq 3 \) and \( r = 2, t \geq 6, \) and can be checked manually for the cases \( r = 2, t \in \{4, 5\} \). Clearly, we can only have equality in (5.12) if \( \tilde{s} = s - 1 \), since otherwise \( \tilde{s} < s - 1 \leq (t - 2)/2 \). Furthermore, \( \tilde{s} < s \) implies
\[ r^{\tilde{s} + 1} \leq (r - 1)t + 1 \]
which gives us
\[ \frac{r^{\tilde{s} + 1} - 1}{r - 1} \leq t = t + 1 - 1 \leq t + 1 - \frac{2\tilde{s}}{t - 2} \] \hspace{1cm} (5.13)
Multiplying by \( t - 2 \) yields
\[ (t - 2) \left( \frac{r^{\tilde{s} + 1} - 1}{r - 1} \right) \leq 2 \left( \left( \frac{t}{2} \right) - \tilde{s} \right) - 2 \]
which is equivalent to
\[ (t - 2) \left( \frac{r^{\tilde{s} + 1} - 1}{r - 1} \right) \leq 2 \left( \left( \frac{t}{2} \right) - \tilde{s} \right) - r^{\tilde{s}}(t - 2) - 2 \]
This finally gives us
\[ (t - 2) \left[ r^{\tilde{s}} \left( \left( \frac{t}{2} \right) - \tilde{s} \right) + \frac{r^{\tilde{s} - 1}}{r - 1} \right] = \left( \left( \frac{t}{2} \right) - \tilde{s} \right) r^{\tilde{s}}(t - 2) + (t - 2) \left( \frac{r^{\tilde{s} - 1}}{r - 1} \right) \]
\[ \leq \left( \left( \frac{t}{2} \right) - \tilde{s} \right) r^{\tilde{s}}(t - 2) + 2 \left( \left( \frac{t}{2} \right) - \tilde{s} \right) - r^{\tilde{s}}(t - 2) - 2 \]
\[ = \left( \left( \frac{t}{2} \right) - \tilde{s} - 1 \right) (r^{\tilde{s}}(t - 2) + 2) \]
This establishes (5.11) and finishes the first part of the proof. For the second part we assume \( \tilde{s} = s \). This yields \( r^{\tilde{s} + 1} > (r - 1)t + 1 \) or stated differently
\[ \frac{r^{\tilde{s} + 1} - 1}{r - 1} > t \]
Since the left hand side (which equals \( 1 + r + \ldots + r^{\tilde{s}} \)) and \( t \) are both integers we get
\[ \frac{r^{\tilde{s} + 1} - 1}{r - 1} \geq t + 1 > t + 1 - \frac{2\tilde{s}}{t - 2} \]
Continuing the calculations as in the first part of the proof from (5.13) onwards yields the converse of (5.11) as claimed.

**Lemma 57.** For all \( \vec{S} \subseteq E(F) \) with \( |\vec{S}| \leq s \) the graph \( F_{\vec{S}} \) is balanced.
Proof. We need to show that for all $D \subseteq F_{\bar{S}}$ we have $d(D) = e(D)/v(D) \leq e(F_{\bar{S}})/v(F_{\bar{S}}) = d(F_{\bar{S}})$. Let $(e_1, e_2, \ldots, e_s) := \bar{S}$. The graph $F_{\bar{S}}$ consists of $r$ copies of $F$ that overlap in the edge $e_1$, $r^2 - r$ copies of $F \setminus \{e_1\}$ each of which shares its edge $e_2$ with 1 copy of $F$ and $r - 2$ other copies of $F \setminus \{e_1\}$, and so on. For all $1 \leq i \leq s$ let $F_{\bar{S}_i}$ denote the set of graphs that contains all such copies of $F \setminus \{e_1, e_2, e_{i-1}\}$ in $F_{\bar{S}}$. We thus obtain

$$d(F_{\bar{S}}) = \frac{1 + \sum_{i=1}^{s} \sum_{H \in F_{\bar{S}_i}} (e(H) - 1)}{2 + \sum_{i=1}^{s} \sum_{H \in F_{\bar{S}_i}} (v(H) - 2)} ,$$

and

$$d(D) = \frac{1 + \sum_{i=1}^{s} \sum_{H \in F_{\bar{S}_i} \cap H \neq \emptyset} (e(D \cap H) - 1)}{2 + \sum_{i=1}^{s} \sum_{H \in F_{\bar{S}_i} \cap H \neq \emptyset} (v(D \cap H) - 2)} .$$

Note that for all $H \in F_{\bar{S}_i}$ we have

$$\frac{e(H) - 1}{v(H) - 2} \geq \frac{e(D \cap H) - 1}{v(D \cap H) - 2} ,$$

since $H$ is 2-balanced by Lemma 55. By Proposition 47 this also implies

$$\frac{e(H) - e(D \cap H)}{v(H) - v(D \cap H)} \geq \frac{e(H) - 1}{v(H) - 2} . \tag{5.14}$$

Furthermore, for all $H' \in F_{\bar{S}_{i+j}}$ for some $j > i$ we have

$$\frac{e(H) - 1}{v(H) - 2} \geq \frac{e(H') - 1}{v(H') - 2} \geq \frac{e(D \cap H') - 1}{v(D \cap H') - 2} .$$

By Lemma 56 we also have

$$\frac{e(H) - 1}{v(H) - 2} \geq \frac{1 + \sum_{j<i} \sum_{H \in F_{\bar{S}_j}} (e(H') - 1)}{2 + \sum_{j<i} \sum_{H \in F_{\bar{S}_j}} (v(H') - 2)} .$$

Alltogether, we obtain

$$\frac{\sum_{H \in F_{\bar{S}_i}} (e(H) - e(D \cap H))}{\sum_{H \in F_{\bar{S}_i}} (v(H) - v(D \cap H))} \geq \frac{\sum_{H \in F_{\bar{S}_i}} (e(H) - 1)}{\sum_{H \in F_{\bar{S}_i}} (v(H) - 2)} \geq \frac{1 + \sum_{j<i} \sum_{H \in F_{\bar{S}_j}} (e(H) - 1)}{2 + \sum_{j<i} \sum_{H \in F_{\bar{S}_j}} (v(H) - 2)} .$$

Consecutively using this inequality for all $1 \leq i \leq s$ in connection with Proposition 47 results in

$$d(D) \leq \frac{1 + \sum_{i=1}^{s} \sum_{H \in F_{\bar{S}_i} \cap H \neq \emptyset} (e(D \cap H) - 1)}{2 + \sum_{i=1}^{s} \sum_{H \in F_{\bar{S}_i} \cap H \neq \emptyset} (v(D \cap H) - 2)} \leq d(F_{\bar{S}}) .$$

$\square$
Corollary 54 follows immediately from Theorem 53 together with the following lemma.

**Lemma 58.** For any \( r \geq 1 \) and \( t \geq 4 \) and \( F := K_t \) we have

\[
\min_{\vec{S} \subseteq E(F), m_2(F \setminus \vec{S}) \leq m(F_{\vec{S}})} m(F_{\vec{S}}) = \theta(t, r) \; .
\]

**Proof.** For all \( \vec{S} \subseteq E(F) \) with \( |\vec{S}| = s \) we have \( m_2(F \setminus \vec{S}) < m(F_{\vec{S}}) \) by Lemma 56 and \( m(F_{\vec{S}}) = d(F_{\vec{S}}) = \theta(t, r) \) by Lemma 57. Clearly, if \( |\vec{S}| > s \) then we have \( m(F_{\vec{S}}) \geq \theta(t, r) \) (in fact we have equality since \( m_2(F \setminus \vec{S}) < m(F_{\vec{S}}) \) for all such \( \vec{S} \)).

If \( |\vec{S}| < s \), Lemma 56 states that we either have \( m_2(F \setminus \vec{S}) > m(F_{\vec{S}}) \) in which case we are done or \( m_2(F \setminus \vec{S}) = m(F_{\vec{S}}) \) in which case we have \( |\vec{S}| = s - 1 \) and furthermore

\[
m(F_{\vec{S}}) = \frac{e(F_{\vec{S}})}{v(F_{\vec{S}})}
= \frac{e(F_{\vec{S}}) + r^{s-1}(e(F \setminus \vec{S}) - 1)}{v(F_{\vec{S}}) + r^{s-1}(v(F \setminus \vec{S}) - 2)}
= \theta(t, r) \; .
\]

\( \square \)
Chapter 6

Outlook

In this chapter we discuss some open questions related to the balanced online graph-avoidance edge coloring game.

6.1 The Asymptotic Gap between Lower and Upper Bound

In this thesis we proved an asymptotic threshold phenomenon for the duration of $\text{OEC}_b(F, r, n)$ for a variety of graphs, including cycles and cliques if the number of colors is sufficiently large. We also showed a general asymptotic upper bound on the duration of $\text{OEC}_b(F, r, n)$ for arbitrary graphs and an arbitrary number of colors, and determined concrete value for this bound for cliques of arbitrary size. For a variety of graphs our results leave an asymptotic gap between the lower bound and the upper bound which calls for further investigation to solve the problem in full generality.

In some sense it seems easier to further improve on the lower bound. The strategy we present in Chapter 4 is quite simple. It just cares about a single subgraph of $F$, namely, the one that can be avoided longest. In fact, it seems that there is room for improvement whenever the condition of the upper bound is not satisfied, i.e., if there is no subgraph $F_\subset F$ with $e(F) - 1$ edges such that $m_2(F_\subset) \leq m_2(F)$. Since the general upper bound from Chapter 5 in this case proceeds over many rounds to obtain a monochromatic copy of $F$, it seems plausible that considering these rounds might also help in the lower bound. This suggests to not only avoid one specific subgraphs of $F$ in the strategy, but also avoid further subgraphs.
Chapter 6. Outlook

Figure 6.1: A situation that might cause our strategy to fail

The Example $F = K_4, r = 2$

For $F = K_4$ and $r = 2$, Theorem 40 proves an asymptotic lower bound of $n^{2-6/11}$ steps. The following strategy a.a.s. survives longer in the graph process: Let $K_{4-}$ denote an arbitrary subgraph of $K_4$ with $e(K_4) - 1 = 5$ edges, and $K_{4-2} \subseteq K_{4-}$ denote an arbitrary subgraph with $e(K_4) - 2 = 4$. In each step, choose an arbitrary coloring that does not close a monochromatic copy of $K_4$, and if possible also does not create a monochromatic copy of $K_{4-}$, otherwise stop. If Painter fails while using this strategy, then there exist two monochromatic copies of $K_{4-}$ one of which she is forced to close by coloring the edges of a certain step. Moreover, each of these two copies of $K_{4-}$ was created in a previous step, in which Painter was confronted with two monochromatic copies of $K_{4-2}$ one of which she had to close. In fact, one can show that this strategy a.a.s. survives for $n^{2-10/19}$, where $19/10$ is the edge density of the graph in Figure 6.1 which illustrates a situation that might cause the strategy to fail. Note that this matches the upper bound proven in Theorem 53 and thus results in a threshold phenomenon for $K_4$ and 2 colors.

While it is easy to check all possibilities for the strategy to fail for $F = K_4$ and $r = 2$ manually, it is still open to prove an improved general lower bound with such a strategy.

6.2 Avoiding Subgraphs in Achlioptas Processes

Krivelevich et al. recently investigated a graph-avoidance game in which seems to be strongly related to OEC$_b(F,r,n)$. They considered the game where in each step the player is presented a set of $r$ edges which is drawn uniformly at random, out of which she has to choose a single one that is added to the graph while the other $r-1$ edges return into the pool of the remaining edges. Such a graph process is called Achlioptas process with parameter $r$. Note that every upper bound on the duration of this game is also an upper bound for OEC$_b(F,r,n)$ and vice versa for lower bounds. Krivelevich et al. proved thresholds on the duration of the Achlioptas game for arbitrary Achlioptas parameters $r$ and certain graphs $F$, namely cliques, cycles of arbitrary size and complete
bipartite graphs with equally large partition classes:

**Theorem 59** ([10]).

(i) For \( t \geq 3 \), the threshold for the online \( C_t \)-avoidance Achlioptas game with parameter \( r \geq 2 \) is

\[
\frac{n^{2-\frac{(t-2)\cdot s+2}{t-1}}}{n^{2-\frac{(t-2)\cdot s+2}{t-1}}}
\]

where \( C_t \) denotes the cycle of length \( t \).

(ii) For \( t \geq 4 \), the threshold for the online \( K_t \)-avoidance Achlioptas game with parameter \( r \geq 2 \) is \( n^{2-\theta} \) where \( K_t \) denotes the complete graph on \( t \) vertices and \( \theta \) is defined as follows:

\[
s := \lfloor \log_r [(r-1)t + 1] \rfloor, \quad \theta = \frac{r^s(t - 2) + 2}{r^s(t \cdot \frac{t}{2} - 1) + \frac{r^s - 1}{r - 1}}.
\]

(iii) For \( t \geq 3 \), the threshold for the online \( K_{t,t} \)-avoidance Achlioptas game with parameter \( r \geq 2 \) is \( n^{2-\theta} \) where \( K_{t,t} \) denotes the complete bipartite graph with partition classes of size \( t \) and \( \theta \) is defined as follows:

\[
s := \lfloor \log_r [(r-1)t + 1] \rfloor, \quad \theta = \frac{r^s(2t - 2) + 2}{r^s(t^2 - 1) + \frac{r^s - 1}{r - 1}}.
\]

Note that their result for cliques matches our result from Corollary 54. Nevertheless, it is still open what the exact connection between the two games is. It would also be interesting to know how the Achlioptas game behaves for arbitrary graphs, since this would probably also be of major interest to understand \( OEC_b(F, r, n) \) in full generality.
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