Ramsey Properties of Random Hypergraphs: the 0-Statement for Cliques

Henning Thomas

ETH Zurich

joint work with
Luca Gugelmann, Yury Person, and Angelika Steger
Ramsey’s theorem

**Theorem (Ramsey 1930)**

Let $k \geq 2$, $r \geq 2$ be fixed. Then there exists a minimum integer $R(k, r)$ such that every $r$-edge-coloring of $K_{R(k,r)}$ contains a monochromatic $K_k$. 

![Graph Illustration](image_url)
Ramsey’s theorem

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**Notation**

We write $G \rightarrow (F)^2_r$ if every $r$-edge-coloring of $G$ contains a monochromatic copy of $F$. 
Randomizations of Ramsey’s Theorem

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color $G(n, p)$ instead $K_n$
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color $G(n, p)$ instead $K_n$
solved completely by
Rödl, Ruciński ’93 &’95
Ramsey’s Theorem in $G(n, p)$

Let $m_2(F) = \max_{J \subseteq F} e(J) - 1 - v(J) - 2$.

The expected number of copies of $F$ on a fixed edge is

$$\Theta(n v(F) - 2p e(F) - 1)$$

if $p \leq cn - v(F) - 2e(F) - 1$.

If $p \geq Cn - v(F) - 2e(F) - 1$, then

$$\Theta(n^2)$$
Ramsey’s Theorem in $G(n, p)$
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The expected number of copies of $F$ on a fixed edge is

$$\Theta(n v(F) - 2 pe(F) - 1) =
\begin{cases}
	iny, e.g. < 0.001 & \text{if } p \leq cn - v(F) - 2 e(F) - 1 \\
\text{huge}, > 10^{10} & \text{if } p \geq Cn - v(F) - 2 e(F) - 1
\end{cases}$$

Let $m_2(F) = \max J \subseteq F e(J) - 1 v(J) - 2 = e(F) - 1 v(F) - 2$. 

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- The expected number of copies of $F$ on a fixed edge is

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Ramsey’s Theorem in $G(n, p)$

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- Let $m_2(F) = \max_{J \subseteq F} \frac{e(J)-1}{v(J)-2}$.

![Diagram of a hypergraph $F$ embedded in a larger hypergraph $G(n,p)$]
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\[ F \quad G(n,p) \]
The Result of Rödl and Ruciński

Let $r \geq 2$ and $F$ be a fixed graph that is not a forest of stars and paths of length 3. Then there exist positive constants $c = c(F, r)$ and $C = C(F, r)$ such that

$$\lim_{n \to \infty} \Pr[\mathcal{G}(n, p) \to (F)^2] = \begin{cases} 1 & \text{if } p \geq \frac{c}{m^2(F)}n - 1 \\ 0 & \text{if } p \leq \frac{c}{m^2(F)}n - 1 \end{cases}$$

For every star $S_k$ the threshold is at

$$p \approx \frac{n - 2}{k^2 - 1} \ll \frac{1}{m^2(S_k)}.$$
The Result of Rödl and Ruciński

**Theorem (Rödl, Ruciński ’93 & ’95)**

Let \( r \geq 2 \) and \( F \) be a **fixed graph** that is not a forest of stars and paths of length 3. Then there exist positive constants \( c = c(F, r) \), and \( C = C(F, r) \) such that

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\lim_{n \to \infty} \Pr[ G(n, p) \rightarrow (F)_{r}^{2} ] = \begin{cases} 
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m_2(P_3) = 1 \quad \text{and} \quad m(S) = 1
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where \( m(F) = \max_{J \subseteq F} \frac{e(J)}{v(J)} \).
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Related Work

- Sharp Thresholds
  - Friedgut, Krivelevich ('00)
  - Friedgut, Rödl, Ruciński, Tetali ('06)

- Asymmetric Ramsey Properties
  - Kohayakawa, Kreuter ('97)
  - Marciniszyn, Spöhel, Skokan, Steger ('09)

- Online Ramsey Games
  - Friedgut, Kohayakawa, Rödl, Ruciński, Tetali ('03)
  - Marciniszyn, Spöhel, Steger '09
  - Marciniszyn, Mitsche, Stojakovic ('07)

- Ramsey Properties of Random Hypergraphs
  - later more...
Related Work

Sharp Thresholds

- Friedgut, Krivelevich ('00) trees
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- Marciniszyn, Mitsche, Stojakovic ('07) balanced, cycles
- Spöhel, Prakash, T. ('09) balanced, large class of graphs
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**Ramsey Properties of Random Hypergraphs**
- later more ...
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(by Allen et al. ’11)
random restriction set \(R_F(n, q)\): flip \(q\)-biased coin for every copy of \(F\) in \(K_n\)
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Combining both Randomizations

The expected number of bad copies of $F$ on a fixed edge is $\Theta(n^{v(F)} - 2p^{e(F)} - 1q)$. 

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Combining both Randomizations

The expected number of bad copies of $F$ on a fixed edge is $\Theta(n^{-v(F)} - 2p + 1q)$. 

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The expected number of bad copies of $F$ on a fixed edge is

$$\Theta(n^{v(F)-2} p^{e(F)-1} q)$$.
Combining both Randomizations

Theorem (Gugelmann, Person, Steger, T. ’12)

Let $r \geq 2$ and $F$ be a strictly 2-balanced graph. Then there exist constants $c = c(F, r) > 0$ and $C = C(F, r) > 0$ such that

$$\lim_{n \to \infty} \Pr\left[ G(n, p) \xrightarrow{\mathcal{R}_F(n,q)} (F)^2_r \right] = \begin{cases} 0, & \text{if } p \leq c n^{v(F)-2} p^{e(F)-1} q \\ 1, & \text{if } p \geq C n^{v(F)-2} p^{e(F)-1} q \end{cases}.$$
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**Proof.** Uses a couple of new ideas . . . later more . . .
$H^{(k)}(n, p)$ denotes a **binomial random $k$-uniform hypergraph**: each of the possible $\binom{n}{k}$ hyperedges appears ind. with prob. $p$. 

The expected number of copies of $F$ on a fixed hyperedge is $\Theta(n v(F) - k p e(F) - 1)$. Is the threshold at $p \approx n^{-1/m_k(F)}$, where $m_k(F) = \max_{J \subseteq F} e(J) - 1 v(J) - k$?
$H^{(k)}(n, p)$ denotes a **binomial random $k$-uniform hypergraph**: each of the possible $\binom{n}{k}$ hyperedges appears ind. with prob. $p$. 

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Is the threshold at $p \approx n^{-1/m_{k}(F)}$, where $m_{k}(F) = \max_{J \subseteq F} e(J) - 1 - v(J) - k$?
$H^{(k)}(n, p)$ denotes a binominal random $k$-uniform hypergraph: each of the possible $\binom{n}{k}$ hyperedges appears ind. with prob. $p$. 

Is the threshold at $p \approx \frac{n - 1}{m_k(F)}$, where $m_k(F) = \max_{J \subseteq F} e(J) - 1 - v(J) - k$?
\( H^{(k)}(n, p) \) denotes a binomial random \( k \)-uniform hypergraph: each of the possible \( \binom{n}{k} \) hyperedges appears ind. with prob. \( p \).
Ramsey in Random Hypergraphs

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$H^{(k)}(n, p)$ denotes a binomial random $k$-uniform hypergraph: each of the possible $\binom{n}{k}$ hyperedges appears ind. with prob. $p$.

The expected number of copies of $F$ on a fixed hyperedge is

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Is the threshold at

$$p \asymp n^{-1/m_k(F)}$$

where $m_k(F) = \max_{J \subseteq F} \frac{e(J) - 1}{v(J) - k}$?
Rödl, Ruciński ('98) confirmed the 1-statement for $F = K_4^{(3)}$. 

Rödl, Ruciński, Schacht ('07) confirmed the 1-statement for $k$-partite, $k$-uniform hypergraphs.

Friedgut, Rödl, Schacht ('10) and, independently, Conlon, Gowers ('12+) confirmed the 1-statement for every $k$-uniform hypergraph.

A corresponding 0-statement is still open.
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Is $p \asymp n^{-1/m_k(F)}$ the right threshold?
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Consider $F = P_3^{(3)}$. 

![Diagram of a hypergraph]
Is $p \preceq n^{-1/m_k(F)}$ the right threshold?

- Consider $F = P_3^{(3)}$. We have $m_k(F) = \frac{3-1}{5-3} = 1$. 

\begin{center}
\includegraphics[width=0.3\textwidth]{hypergraph.png}
\end{center}
Is $p \approx n^{-1/m_k(F)}$ the right threshold?

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[Diagram of a hypergraph]
Is $p \gtrapprox n^{-1/m_k(F)}$ the right threshold?

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![Hypergraph diagram](image)
Is $p \approx n^{-1/m_k(F)}$ the right threshold?

- Consider $F = P_3^{(3)}$. We have $m_k(F) = \frac{3-1}{5-3} = 1$.

- Consider the following hypergraph $H$. 

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- Consider $F = P_3^{(3)}$. We have $m_k(F) = \frac{3-1}{5-3} = 1$.

- Consider the following hypergraph $H$. We have $m(H) = \frac{10}{11} < 1$. 
Is $p \approx n^{-1/m_k(F)}$ the right threshold?

- Consider $F = P_3^{(3)}$. We have $m_k(F) = \frac{3-1}{5-3} = 1$.

- Consider the following hypergraph $H$. We have $m(H) = \frac{10}{11} < 1$.

- $H$ appears in $H^{(k)}(n, p)$ if $p = cn^{-1/m_k(F)} = cn^{-1}$.
Is $p \asymp n^{-1/m_k(F)}$ the right threshold?

- Consider $F = P_3^{(3)}$. We have $m_k(F) = \frac{3-1}{5-3} = 1$.

- Consider the following hypergraph $H$. We have $m(H) = \frac{10}{11} < 1$.

- $H$ even appears in $H^{(k)}(n, p)$ if $p = cn^{-1/m_k(F)-0.1} = cn^{-1.1}$. 

Is $p \asymp n^{-1/m_k(F)}$ the right threshold?

- Consider the following 4-uniform hypergraph $F$. 
Is $p \propto n^{-1/m_k(F)}$ the right threshold?

Consider the following 4-uniform hypergraph $F$.

We have $m_4(F) = \frac{3-1}{5-4} = 2$. 
Is \( p \asymp n^{-1/m_k(F)} \) the right threshold?

- Consider the following 4-uniform hypergraph \( F \).

We have \( m_4(F) = \frac{3-1}{5-4} = 2 \).
Is \( p \ll n^{-1/m_k(F)} \) the right threshold?

- Consider the following 4-uniform hypergraph \( F \).
  
  ![Graph 1]

  We have \( m_4(F) = \frac{3-1}{5-4} = 2 \).

- Consider the following hypergraph \( H \).
  
  ![Graph 2]

  \( K_6 \)
Is $p \sim n^{-1/m_k(F)}$ the right threshold?

- Consider the following 4-uniform hypergraph $F$.

  ![Hypergraph F](image)

  We have $m_4(F) = \frac{3-1}{5-4} = 2$.

- Consider the following hypergraph $H$.

  ![Hypergraph H](image)

  We have $m(H) = \frac{15}{8} < 2$. 

In general, \( p \sim n^{-1/m_k(F)} \) is not the right threshold, but
In general, $p \asymp n^{-1/m_k(F)}$ is not the right threshold, but

**Theorem (Gugelmann, Person, Steger, T. ’12)**

Let $k \geq 3$, $\ell > k$ and let $F = K_{\ell}^{(k)}$. Then there exists a constant $c = c(k, \ell) > 0$ such that

$$\lim_{n \to \infty} \Pr[H^{(k)}(n, p) \to (F)^2] = 0 \quad \text{if } p \leq cn^{-1/m_k(F)}.$$
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\]

**Proof.**

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**Proof.**

- Fix $k \geq 3$ and $\ell > k$ and let $F = K_{\ell}^{(k)}$.
- Assume that $p \leq cn^{-1/m_k(F)}$.
- We need to show that we can a.a.s. color the hyperedges of $H^{(k)}(n, p)$ with 2 colors without a monochromatic copy of $F$. 
We call a hyperedge of a hypergraph $H$ closed (in $H$) if it is contained in at least 2 otherwise edge-disjoint copies of $F$ in $H$. We call a hypergraph $H$ closed if all its edges are closed in $H$.

Idea. Successively remove open edges of $H$ ($k(n,p)$). We are left with a collection of closed subhypergraphs which remain to be colored.
We call a hyperedge of a hypergraph $H$ **closed** (in $H$) if it is contained in at least 2 otherwise edge-disjoint copies of $F$ in $H$. It is **open** (in $H$) otherwise.

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**Idea.** Successively remove open edges of $H$ ($k(n, p)$). We are left with a collection of closed subhypergraphs which remain to be colored.
Open and Closed Edges

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![Diagram showing closed hyperedges](image)
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![Diagram of a hyperedge being contained within another hyperedge with a checkmark indicating a closed edge.](image)
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![Diagram showing a hypergraph with a closed edge highlighted]
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**Idea.** Successively remove open edges of $H^{(k)}(n, p)$.

We are left with a collection of closed subhypergraphs which remain to be colored.
Proof Outline

There exists a constant $L > 0$ such that a.a.s. every closed subhypergraph has size at most $L$.

(D) For every closed hypergraph $H$ with $m(H) \leq mk(F)$ we have $H \not \rightarrow (F)_k^2$.

Note that (P) implies by small subgraphs that a.a.s. every constant size closed subhypergraph $H$ in $H(k)(n,p)$ satisfies $m(H) \leq mk(F)$. 

Henning Thomas (ETH Zurich)  Ramsey Properties of Random Hypergraphs  June 26, 2012  16 / 21
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**P** There exists a constant \( L > 0 \) such that a.a.s. every closed subhypergraph has size at most \( L \). \( \text{(works if} \ \delta(F) \geq 2\text{)} \)

**D** For every closed hypergraph \( H \) with \( m(H) \leq m_k(F) \) we have \( H \nrightarrow (F)_2^k \).

Note that **P** implies by small subgraphs that a.a.s. every constant size closed subhypergraph \( H \) in \( H^{(k)}(n, p) \) satisfies \( m(H) \leq m_k(F) \).
Proof Outline

(P) There exists a constant \( L > 0 \) such that a.a.s. every closed subhypergraph has size at most \( L \). \((\text{works if } \delta(F) \geq 2)\)

(D) For every closed hypergraph \( H \) with \( m(H) \leq m_k(F) \) we have \( H \not\rightarrow (F)_2^k \). \((\text{works for a large class of graphs})\)

Note that (P) implies by small subgraphs that a.a.s. every constant size closed subhypergraph \( H \) in \( H^{(k)}(n,p) \) satisfies \( m(H) \leq m_k(F) \).
Building closed subhypergraphs
(P) Building closed subhypergraphs
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Structure becomes large and unlikely to appear
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Structure becomes dense and unlikely to appear
(D) Special Case: Cliques

Need to show: we can color the edges of every hypergraph $H$ with $m(H) \leq m_k(K_k \ell)$ without a monochromatic copy of $K_k \ell$.

Implies for every $H' \subseteq H$ that $\delta(H') \leq km_k(K_k \ell)$.

Color $H$ iteratively.

Suffices to avoid a $K_{k-1} \ell - 1$ in the link of $v$.

$km_k(K_k \ell)$ hyperedges do not enforce a monochromatic $K_{k-1} \ell - 1$. 
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Henning Thomas (ETH Zurich)

Ramsey Properties of Random Hypergraphs

June 26, 2012 19 / 21
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  ![Diagram](image)

- **Color $H$ iteratively**

  ![Diagram](image)

- **Suffices to avoid a $K^{(k-1)}_{\ell-1}$ in the link of $v$**
(D) Special Case: Cliques

- **Need to show**: we can color the edges of every hypergraph $H$ with $m(H) \leq m_k(K_{\ell}^{(k)})$ without a monochromatic copy of $K_{\ell}^{(k)}$.
- ...implies for every $H' \subseteq H$ that $\delta(H') \leq km_{k}(K_{\ell}^{(k)})$.

![Diagram showing a hypergraph]

- Color $H$ iteratively

  ![Diagram showing a hypergraph]

- Suffices to avoid a $K_{\ell-1}^{(k-1)}$ in the link of $v$
- $km_{k}(K_{\ell}^{(k)})$ hyperedges do not enforce a monochromatic $K_{\ell-1}^{(k-1)}$. 
Theorem (Gugelmann, Person, Steger, T. '12)

Let \( k \geq 3 \) and \( F \) be a \( k \)-uniform hypergraph with \( m_k(F) \geq 1 \). If \( F \) contains a strictly \( k \)-balanced subgraph \( F' \) with \( m_k(F') = m_k(F) \) and \( F' \) is a clique, or \( \chi(F') \geq k + 1 \), or \( m(F') \geq 2 + 2^k - 1 v(F') - 2^k \), or \( F' \) is spacious, or \( \lfloor km_k(F') \rfloor \leq r(\delta(F') - 1) \) or \( \lfloor km_k(F') \rfloor < (\chi(F') - 1)r \), then there exists a constant \( c = c(F) > 0 \) such that \( \lim_{n \to \infty} \Pr[H(k)(n, p) \to F] = 0 \) if \( p \leq cn^{-1/m_k(F)} \).
Our Full Result

Theorem (Gugelmann, Person, Steger, T. ’12)

Let $k \geq 3$ and $F$ be a $k$-uniform hypergraph with $m_k(F) \geq 1$. If $F$ contains a strictly $k$-balanced subgraph $F'$ with $m_k(F) = m_k(F')$ and

- $F'$ is a clique, or
- $\chi(F') \geq k + 1$, or
- $m(F') \geq 2 + \frac{2k-1}{v(F')-2k}$, or
- $F'$ is spacious, or
- $\lfloor km_k(F') \rfloor \leq r(\delta(F') - 1)$ or $\lfloor km_k(F') \rfloor < (\chi(F') - 1)^r$,

then there exists a constant $c = c(F) > 0$ such that

$$\lim_{n \to \infty} \Pr[H^{(k)}(n, p) \to (F)^k_2] = 0 \quad \text{if } p \leq cn^{-1/m_k(F)}.$$
Open Problems

For which hypergraphs \( F \) do we have
\[
 m(H) \leq m_k(F) \Rightarrow H \not\rightarrow (F)^k_2
\]

What is the threshold for all other hypergraphs?
Open Problems

- **Full Characterization** for which hypergraphs the threshold is at
  \[ p \gtrsim n^{-1/m_k(F)} \]
Open Problems

- **Full Characterization** for which hypergraphs the threshold is at
  
  \[ p \approx n^{-1/m_k(F)} \]

- For which hypergraphs \( F \) do we have

  \[ m(H) \leq m_k(F) \Rightarrow H \not\rightarrow (F)_2^k \]
Open Problems

- **Full Characterization** for which hypergraphs the threshold is at 
  \( p \lesssim n^{-1/m_k(F)} \)

- For which hypergraphs \( F \) do we have 
  \[ m(H) \leq m_k(F) \implies H \leftrightarrow (F)_2^k \]?

- What is the threshold for all other hypergraphs?