On Kontsevich’s Formality Theorem

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Abstract

This paper is submitted as a dissertation for the degree of a Master of Science in Mathematics. It reviews the link between formal deformation quantizations and Poisson manifolds, which was conjectured in 1993. This link is a corollary of a more general statement called "Formality Theorem", proved by Maxim Kontsevich in 1997.

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2 Introduction

In 1997, Maxim Kontsevich published a paper about formal deformation quantization of Poisson manifolds [Kon]. His paper attracted a lot of attention, because he applied some ideas from string theory, which hadn’t been used in this context before.

This paper aims to give an impression of how these methods are used. So it essentially reviews Kontsevich’s original proof of his so-called "Formality Theorem". However, it doesn’t always give full details, but rather tries to ignore technicalities and concentrate on presenting the new means.

The structure of this paper is as follows: Chapter 3 tries to give some motivation by explaining the link between Poisson manifolds and formal deformation quantization, which is perhaps the most important application of the formality theorem.

Chapter 4 is dedicated to formulating the formality theorem. It includes some necessary algebraic preparations.

The next two chapters outline Kontsevich’s proof. While chapter 5 presents the tools that are used, the action takes place in chapter 6. In section 6.1, a morphism between certain spaces is constructed, and in 6.2 it is shown that this morphism has all the required properties.

Finally, Chapter 7 tries to show why Kontsevich’s work is so important for this field of mathematics. It also points out further developments that have been achieved as a result of Kontsevich’s paper. A reader who wants to learn even more about this topic will find some helpful references in this chapter.
3 Motivation and Applications of the Formality Theorem

3.1 Multiplication on Poisson Algebras

One of the main applications (and one of the most important motivations for the whole theory) is the deformation quantization problem. In order to explain this problem we first need the definitions of a formal deformation and of a Poisson manifold:

**Definition 3.1** Let $A$ be an associative algebra over a field of characteristic zero. By $A[[t]]$ we denote the space of formal power series in a variable $t$ with coefficients in $A$. A formal deformation of $A$ is an associative $k[[t]]$-bilinear multiplication law $m : A[[t]] \otimes_{k[[t]]} A[[t]] \to A[[t]]$, such that for $a, b \in A$

$$m(a, b) = ab + \sum_{i=1}^{\infty} m_i(a, b) \cdot t^i.$$ 

By $ab$ we denote the original multiplication on $A$.

Note that the associativity of $m$ can formally be expressed as

$$m(m(a, b), c) - m(a, m(b, c)) = 0 \tag{1}$$

for all $a, b, c \in A$.

**Definition 3.2** An associative algebra $A$ is called a (non-commutative) Poisson algebra, if it comes along with a bracket $\{\cdot, \cdot\} : A \otimes A \to A$ which satisfies the following rules:

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$$

$$\{a, b\} - \{b, a\} = 0$$

for all $a, b, c \in A$

These two structures are naturally linked. Given a formal deformation its first-order part gives rise to a Poisson bracket

$$\{a, b\} := \frac{1}{2}(m_1(a, b) - m_1(b, a)).$$

The Jacobi identity holds because of the cancellation of terms like $m_1(a, m_1(b, c)) - m_1(m_1(a, b), c)$, so it is a direct consequence of the associativity of $m$. 
The deformation quantization problem is the reverse problem: Given a Poisson algebra $A$, find a fitting formal deformation which respects the Poisson structure, i.e. the commutator of the first-order part of the deformation should be the original Poisson structure on $A$.

The crucial point about such a deformation is that it has to be associative, so it has to satisfy equation (1). The left hand side of this equation turns out to be a special case of the so-called Gerstenhaber bracket (also simply known as G-bracket):

**Definition 3.3** Let $\phi_i \in C^{k_i}(A, A) := \text{Hom}(\otimes^{k_i+1} A, A)$, where $k_i \geq -1$ for $i = 1, 2$. Then the Gerstenhaber bracket is defined as

$$[\phi_1, \phi_2] := \phi_1 \circ \phi_2 - (-1)^{(k_1+1)(k_2+1)} \phi_2 \circ \phi_1 \in C^{k_1+k_2+1}(A, A),$$

where the (non-associative) product $\circ$ is defined as

$$(\phi_1 \circ \phi_2)(a_0 \otimes \ldots \otimes a_{k_1+k_2}) := 
\sum_{i=0}^{k_1} (-1)^{k_2} \phi_1(a_0 \otimes \ldots \otimes a_{i-1} \otimes (\phi_2(a_i \otimes \ldots \otimes a_{i+k_2})) \otimes a_{i+k_2+1} \otimes \ldots \otimes a_{k_1+k_2})$$

**Remark 3.4** The Gerstenhaber bracket can also be defined in a more conceptual way: The coalgebra of $A$ can be identified with $\bigoplus_{n \geq 0}(\otimes^n A)$, and so the space of graded derivations of this coalgebra has $k$-th degree part $\text{Hom}(\otimes^k A, A) = C^{k-1}(A, A)$. In this terminology, the Gerstenhaber bracket is just the commutator of derivations, as Stasheff pointed out in [Sta].

However, by using the formula, we can see that the associativity of the the deformation $m(a, b) = \sum_{i=0}^{\infty} m_i(a, b)t^i$ can be expressed as $[m, m] = 0$. This simple fact allows us to apply powerful algebraic means in order to examine the condition of being associative.

Since the original multiplication $m_0$ is associative, we have automatically $[m_0, m_0] = 0$. But this equation yields a differential $d : C^k(A, A) \rightarrow C^{k+1}(A, A)$, the Hochschild differential, which is just the adjoint of $m_0$, so $d\phi := [\phi, m_0]$. Explicitly,

$$d\phi(a_1 \otimes \ldots \otimes a_{n+1}) = a_1 \phi(a_2 \otimes \ldots \otimes a_n) +$$

$$+ \sum_{i=0}^n (-1)^i \phi(a_1 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \ldots \otimes a_n + (-1)^{n+1} \phi(a_1 \otimes \ldots \otimes a_n)a_{n+1}$$

The bracket and the differential turn the Hochschild complex $C(A, A) := \bigoplus_{k \geq -1} C^k(A, A)[-k]$ into a differential graded Lie algebra. (By $[-k]$ we
denote a grading shift by $k$, i.e. $C^k(A, A)[-k]$ contributes only to the $k$-th degree part of $C(A, A)$. The axioms for this structure are easy to check:

$$d^2 = 0$$

$$d[\phi_1, \phi_2] = [d\phi_1, \phi_2] + (-1)^{\phi_1|\phi_2}[\phi_1, d\phi_2]$$

$$[\phi_1, \phi_2] = -(-1)^{|\phi_1||\phi_2|}[\phi_2, \phi_1]$$

$$[\phi_1[\phi_2, \phi_3]] + (-1)^{|\phi_1|(|\phi_2|+|\phi_3|)}[\phi_3[\phi_1, \phi_2]] + (-1)^{|\phi_1|(|\phi_2|+|\phi_3|)}[\phi_2[\phi_3, \phi_1]] = 0$$

In these formulas, $\phi_i$ means a homogeneous element of degree $|\phi_i|$, so $\phi_i \in C_{|\phi_i|}$.

Since $d^2 = 0$, one can form the Hochschild cohomology

$$H^*(A, A) := \text{Ker}(d)/\text{Im}(d),$$

and the shifted Hochschild cohomology $H^*(A, A)[1]$ inherits the structure of a graded Lie algebra.

In this expression, the shifting $[1]$ means the following: If $V$ is a graded vector space then $V[k]$ is the graded vector space with $n$-th degree part $V[k]^n = V^{n-k}$.

Now let us take a closer look at the condition $[m, m] = 0$ (which is equivalent to the associativity of the deformation $m$). If we decompose $m$ into its zero degree part $m_0$ (which is just the usual multiplication of $A$) and higher order terms $m' := m - m_0 = \sum_{i=1}^\infty m_i \cdot t^i$, we obtain:

$$0 = [m, m]$$

$$= [m_0 + m', m_0 + m']$$

$$= [m_0, m_0] + [m_0, m'] + [m', m_0] + [m', m']$$

$$= 2dm' + [m', m'],$$

since $[m_0, m_0] = 0$ as $m_0$ is associative and $[m_0, m'] = -(-1)^{1 \cdot 1}[m', m_0] = dm'$.

The equation $2dm' + [m', m'] = 0$ is called the Maurer-Cartan equation.

In the new terminology, we are looking for a solution $m' = \sum_{i=1}^\infty m_i t^i$ of the Maurer-Cartan equation, such that the commutator of the first-order part $m_1$ is the given Poisson structure.

In [Kon], Kontsevich proved the existence of such a deformation for Poisson manifolds. Recall that a Poisson manifold is a manifold $X$, equipped with a Poisson bracket on the algebra $A = C^\infty(X)$ of smooth functions on $X$. A deformation of a Poisson manifold is a deformation of $A$, the components $m_i$ of which are bidifferential operators.
4 The statement

We have already seen that we can describe deformations as elements of the Hochschild complex. In this chapter, we will define two differential graded Lie algebras $D_{\text{poly}}(X)$ and $T_{\text{poly}}(X)$, containing deformations and Poisson structures, respectively. We will then state Kontsevich’s theorem, which links these Lie algebras in terms of so-called quasi-isomorphisms, and by it solves the deformation quantization problem.

4.1 $D_{\text{poly}}(X)$

**Definition 4.1** Let $A$ be the algebra of smooth functions on $X$. The differential graded Lie algebra $D_{\text{poly}}(X)$ consists of all homomorphisms $A^\otimes n \to A$ that are given by polydifferential operators. Equipped with the Gerstenhaber bracket and the Hochschild differential, $D_{\text{poly}}(X)$ is a differential graded Lie subalgebra of the Hochschild complex.

In a local chart $(x_i)$ all elements of $D_{\text{poly}}(X)$ look like

$$f_0 \otimes \ldots \otimes f_n \to \sum_{I_0,\ldots,I_n} C^{I_0,\ldots,I_n}(x) \cdot \partial_{I_0}(f_0)\ldots\partial_{I_n}(f_n),$$

where the sum is finite, the $I_k$ are multi-indices and $\partial_{I_k}$ are the corresponding partial derivatives.

4.2 $T_{\text{poly}}(X)$

**Definition 4.2** The differential graded Lie algebra $T_{\text{poly}}(X)$ consists of all polyvector fields on $X$. Its $n$-th degree term is:

$$T_{\text{poly}}^n(X) := \Gamma(X, \bigwedge^{n+1} T_X).$$

The differential is just 0, and the bracket, the Schouten-Niehnhuis bracket, is obtained by extending the normal Lie bracket of vector fields via the Leibniz rule.

Explicitly, the formula for the bracket is:

$$[\xi_0 \wedge \ldots \wedge \xi_k, \eta_0, \ldots, \eta_l] =$$

$$= \sum_{i=0}^k \sum_{j=0}^l (-1)^{i+j+k} [\xi_i, \eta_j] \wedge \xi_0 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \xi_k \wedge \eta_0 \wedge \ldots \wedge \hat{\eta}_j \wedge \ldots \wedge \eta_l$$

for $\xi_i, \eta_l \in \Gamma(X, T_X)$.
Let us briefly introduce a new notation for vector fields, and thereby simplify the definition of the bracket. Given a local chart \((x^1, \ldots, x^d)\), we can interpret the vector fields \(\partial/\partial x^i\) as odd variables \(\psi_i\). If we forget about the definition of those \(\psi_i\) and simply treat them as variables, we can derive in direction of those new quantities:

\[
\frac{\partial \alpha_{i_1 \ldots i_n} \psi_{i_1} \cdots \psi_{i_n}}{\partial \psi_{i_k}} = (-1)^{k-1} \alpha_{i_1 \ldots i_n} \psi_{i_1} \cdots \hat{\psi}_{i_k} \cdots \psi_{i_n}.
\]

With this notation, the bracket can be written as the commutator

\[
[\alpha_1, \alpha_2] := \alpha_1 \bullet \alpha_2 - (-1)^{k_1 k_2} \alpha_2 \bullet \alpha_1
\]

of the product

\[
\alpha_1 \bullet \alpha_2 := \sum_{i=1}^d \frac{\partial \alpha_1}{\partial \psi_i} \frac{\partial \alpha_2}{\partial x^i}.
\]

In this definition, \(\alpha_1 \in T^{k_1}_{\text{poly}} X, \alpha_2 \in T^{k_2}_{\text{poly}} X\).

Note that Poisson structures on \(X\) are precisely those elements \(\alpha \in T^1_{\text{poly}}(X)\) that satisfy the equation \([\alpha, \alpha] = 0\), which is exactly the Maurer-Cartan-equation for \(T_{\text{poly}}(X)\) (using the fact that the differential is zero).

### 4.3 Coalgebras

If we could find an isomorphism between the two differential graded Lie algebras \(D_{\text{poly}}(X)\) and \(T_{\text{poly}}(X)\), we would get a link between Poisson structures and formal deformations. Unfortunately, in general these two objects seem not to be isomorphic. However, Kontsevich found a way to cure this. He relaxed the relation of being isomorphic to being quasi-isomorphic. It turns out that this is on the one hand weak enough, so he could actually find an explicit quasi-isomorphism between the two spaces, and on the other hand it was strong enough, so he would still get a sufficient close link between Poisson structures and formal deformations.

In this section we will give some further algebraic background, so that in the next section we have the terminology to specify quasi-isomorphisms.

Recall that a graded coalgebra \(A\) is a graded vector space with a comultiplication \(\Delta : A \to A \otimes A\), which is coassociative, i.e. the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \otimes A \\
\Delta & & \Delta \otimes 1 \\
A \otimes A & \xrightarrow{1 \otimes \Delta} & A \otimes A \otimes A
\end{array}
\]
4.4 \( L_{\infty} \)-algebras

**Definition 4.3** A coalgebra without co-unit, cofree cogenerated by a vector space \( V \), is given by \( C(V) = \bigoplus_{n \geq 1} (\bigotimes^n V)^{\Sigma_n} \), where the exponent \( \Sigma_n \) means symmetrization by the action of the \( n \)-th permutation group. The comultiplication \( \Delta \) of this space is given by

\[
\Delta(v_1 \otimes \ldots \otimes v_n) = \sum_{I \cup J = \{1, \ldots, n\}} \epsilon_v(I, J) v_I \otimes v_J,
\]

where \( \epsilon_v \) contributes to the fact that we have to interchange some of the \( v_k \).

We have been a bit sloppy in the definition of the comultiplication. In fact, the term \( \epsilon_v(I, J) \) makes only sense if \( I \) and \( J \) are ordered sets. In our definition though, we summed over unordered sets. However, the term \( \epsilon_v(I, J) v_I \otimes v_J \) yields the same, even if we change the internal order of \( I \) and \( J \).

The cofree coalgebra without co-unit, cofree cogenerated by a vector space \( V \), can be described in terms of universal algebra: It is the coalgebra \( C(V) \), for which there exists a map \( \pi : C(V) \to V \), such that for every coalgebra \( X \) with a map \( p : X \to V \), the following commuting diagram can be completed (in a unique way):

\[
\begin{array}{ccc}
V & \xrightarrow{\pi} & C(V) \\
| & \downarrow{\phi} & \\
X & \xleftarrow{p} & \end{array}
\]

4.4 \( L_{\infty} \)-algebras

In order to formulate Kontsevich’s statement, it is convenient to introduce the language of \( L_{\infty} \)-algebras and quasi-isomorphisms:

**Definition 4.4** An \( L_{\infty} \)-algebra is a graded vector space \( g \), together with a codifferential \( Q \) (so \( Q \circ Q = 0 \)) of degree 1 on \( C(g) \), which is the cofree cocommutative coassociative coalgebra without counit cogenerated by \( g \).

Using the universal property, we can describe \( Q \) as a function \( C(g) \to g \). But by breaking up \( C(g) \) into its degree parts, such a function can be described via its Taylor coefficients. Those coefficients are linear functions, which together determine \( Q \):

\[
\begin{align*}
Q_1 & : g \to g[1] \\
Q_2 & : \Lambda^2(g) \to g \\
Q_3 & : \Lambda^3(g) \to g[-1] \\
& \vdots
\end{align*}
\]
Let us figure out what the condition $Q \circ Q = 0$ means in terms of these Taylor coefficients. Let us first apply $Q \circ Q$ to an element of the underlying vector space:

$$0 = (Q \circ Q)(v) = Q(Q_1(v_1)) = Q_1(Q_1(v_1))$$

So $Q_1$ is a differential on the vector space.

Let us now tackle the case where the argument is of the form $v_1 \wedge v_2$:

$$Q(v_1 \wedge v_2) = Q_1(v_1) \wedge v_2 + (-1)^{|v_1|} v_1 \wedge Q_1(v_2) + Q_2(v_1 \wedge v_2).$$

$$\Rightarrow 0 = Q(Q(v_1 \wedge v_2))$$

$$= Q_1(Q_1(v_1) \wedge v_2) + (-1)^{|v_1|} Q_1(v_1) \wedge Q_1(v_2) + Q_2(v_1 \wedge v_2)$$

$$= Q_1(Q_1(v_1)) \wedge v_2 + (-1)^{|v_1|} Q_1(v_1) \wedge Q_1(v_2) + Q_2(v_1 \wedge v_2)$$

$$+ (-1)^{|v_1|} v_1 \wedge Q_1(v_2) + (-1)^{|v_1|+|v_2|} v_1 \wedge v_2 \wedge Q_1(v_2) + Q_1(Q_1(v_2))$$

$$= Q_1(v_2) + Q_2(v_1 \wedge v_2) + Q_2(v_1 \wedge v_2)$$

$$= Q_1(Q_2(v_1 \wedge v_2)) + Q_2(Q_1(v_1) \wedge v_2) + (-1)^{|v_1|} Q_2(v_1 \wedge Q_1(v_2)).$$

The last expression reminds us of one of the axioms for a differential graded Lie algebra, where $Q_1$ and $Q_2$ play the roles of the differential and the Lie bracket, respectively. Although it gets a bit messy, let us dare to carry out the calculation for an argument $v_1 \wedge v_2 \wedge v_3$:

$$Q(v_1 \wedge v_2 \wedge v_3) = Q_1(v_1) \wedge v_2 \wedge v_3 + (-1)^{|v_1|} v_1 \wedge Q_1(v_2) \wedge v_3 +$$

$$+ (-1)^{|v_1|+|v_2|} v_1 \wedge v_2 \wedge Q_1(v_3) + Q_2(v_1 \wedge v_2) \wedge v_3 +$$

$$+ (-1)^{|v_1|+|v_2|+|v_3|} Q_2(v_2 \wedge v_3) \wedge v_1 +$$

$$+ Q_3(v_1 \wedge v_2 \wedge v_3)$$

$$\Rightarrow 0 = Q(Q(v_1 \wedge v_2 \wedge v_3))$$

$$= Q_3(Q_1(v_1) \wedge v_2 \wedge v_3) + (-1)^{|v_1|} Q_3(v_1 \wedge Q_1(v_2) \wedge v_3) +$$

$$+ (-1)^{|v_1|+|v_2|} Q_3(v_1 \wedge v_2 \wedge Q_1(v_3)) + Q_2(Q_2(v_1 \wedge v_2) \wedge v_3) +$$

$$+ (-1)^{|v_1|+|v_2|} Q_2(Q_2(v_2 \wedge v_3) \wedge v_1) +$$

$$+ (-1)^{|v_3|(|v_1|+|v_2|)} Q_2(Q_2(v_2 \wedge v_3) \wedge v_2) +$$

$$+ Q_1(Q_3(v_1 \wedge v_2 \wedge v_3))$$

If the third Taylor coefficient $Q_3$ vanishes, this is just the Jacobi identity, so $Q_2$ is then a Lie bracket. In fact every differential graded Lie algebra can be interpreted as a $L_\infty$-algebra with all Taylor coefficient vanishing except for $Q_1 = d$ and $Q_2 = [\cdot, \cdot]$. 


Furthermore, since $Q_1 : g \rightarrow g[1]$ is a differential, the pair $(g, Q_1)$ is in fact a complex.

Note: The homology with respect to $Q_1$ of an $L_\infty$-algebra is a graded Lie algebra. So $L_\infty$-algebras are also referred to as Lie algebras up to homotopy.

Now we have collected enough algebraic facts to define isomorphisms and quasi-isomorphisms of $L_\infty$-algebras:

**Definition 4.5** A $L_\infty$-morphism between two $L_\infty$-algebras $g_1$ and $g_2$ is a morphism $F : C(g_1)[1] \rightarrow C(g_2)[1]$ of graded cocommutative coassociative coalgebras, which is compatible with differentials, i.e. $F \circ Q^1 = Q^2 \circ F$, where $Q^1$ and $Q^2$ are the coderivation of $g_1$ and $g_2$, respectively.

Note that since a $L_\infty$-morphism respects the differential and the bracket, a solution of the Maurer-Cartan equation $2dm + [m, m] = 0$ will stay a solution of this equation under the $L_\infty$-morphism.

Such a $L_\infty$-morphism also induces a morphism of the complexes $(g_i, Q^1_i)$, and hence it induces a morphism in the cohomology of this complex. So we can define:

**Definition 4.6** A quasi-isomorphism between two $L_\infty$-algebras is a $L_\infty$-morphism which induces an isomorphism between the cohomology groups of the corresponding complexes.

The name quasi-isomorphism is justified by the next well-known theorem (e.g. [HS]):

**Theorem 4.7** Let $F$ be a quasi-isomorphism between two $L_\infty$-algebras. Then there exists a $L_\infty$-morphism going in the other direction, which induces the inverse homomorphism between the cohomology groups of the corresponding complexes.

This tells us that the property of being quasi-isomorphic is indeed an equivalence relation of $L_\infty$-algebras.

### 4.5 Kontsevich’s statement

Finally we have enough terminology to formulate the statement that Kontsevich proved in 1997:

**Theorem 4.8** (Kontsevich): The differential graded Lie algebras $T_{\text{poly}}(X)$ and $D_{\text{poly}}(X)$ are quasi-isomorphic.
Before we outline Kontsevich’s proof, let us show why this theorem gives a solution to the deformation quantization problem. We have already done all the work, so we just have to collect the facts:

Given a Poisson manifold $X$, we can represent the Poisson bracket by a bivector field $\alpha \in T^1_{\text{poly}}(X)$. We want to construct a star product $\mu + \gamma$, where $\mu$ is the multiplication on $C^\infty(X)$, and $\gamma \in D_{\text{poly}}(X) \otimes t \mathbb{R}[[t]]$, where $t$ is a formal parameter. Moreover, $\gamma$ is required to be a solution of the Maurer-Cartan equation (This condition makes the product associative).

So we start with the formal path $\alpha t \in T_{\text{poly}}(X) \otimes t \mathbb{R}[[t]]$. Via the quasi-isomorphism, we obtain our $\gamma$. Recall that since $\alpha$ is a Poisson structure, $\alpha t$ is a solution of the Maurer-Cartan-equation. Hence, $\gamma$ is also a solution of this equation. Thus, $\mu + \gamma$ is associative, whereby we have indeed found our star product.

(In this argumentation, we have been a bit inexact. We had only shown that the Maurer-Cartan equation is respected when we pass from $T_{\text{poly}}(X)$ to $D_{\text{poly}}(X)$. Rigidly spoken, we rather want to pass from $T_{\text{poly}}(X) \otimes t \mathbb{R}[[t]]$ to $D_{\text{poly}}(X) \otimes t \mathbb{R}[[t]]$, so we would need that the Maurer-Cartan equation is also invariant under this transformation. A formal proof of this can be found in [AMM].)

5 Preparations for the proof

In this and the next chapter, we will outline Kontsevich’s original proof in the (non-trivial) case where the underlying manifold is $\mathbb{R}^n$. Due to time and space limitations, we will not give all the details but rather try to give an impression of the tools Kontsevich used. In particular, whenever the proof would require to treat many different cases, we will content ourselves with just considering one of them. A complete proof can be found in [Kon] and [AMM].

5.1 A naive approach

Kontsevich’s proof is constructive, i.e. he defines an explicit quasi-isomorphism $U : T_{\text{poly}}(X) \to D_{\text{poly}}(X)$. Here is a first attempt to find such a quasi-isomorphism.

There is a natural map $U_1 : T_{\text{poly}}(X) \to D_{\text{poly}}(X)$:

$$U_1(\xi_0 \wedge ... \wedge \xi_n) := \left( f_1 \otimes \ldots \otimes f_n \mapsto \frac{1}{(n+1)!} \sum_{\sigma \in \Sigma_{n+1}} \text{sgn}(\sigma) \prod_{i=0}^{n} \xi_{\sigma(i)}(f_i) \right).$$

This map induces a quasi-isomorphism of complexes (cf. [Kon]), but it is not an $L_\infty$-morphism, i.e. it does not send the Schouten-Nijenhuis bracket to the Gerstenhaber bracket. However, it is not completely useless. The first
Taylor coefficient of our $L_\infty$-morphism $U$ will turn out to be $U_1$. Thus, $U$ will automatically induce a quasi-isomorphism of complexes, too.

5.2 Configuration spaces

5.2.1 Some definitions

Before we can proceed in approaching $U$, we will need some more tools. This and the next chapters are devoted to introducing these tools.

Let $\mathcal{H} := \{z \in \mathbb{C} | \Im(z) > 0\}$ be the upper half-plane. For non-negative integers $n, m$ satisfying the inequality $2n + m \geq 2$ we define the space

$$Conf_{n,m} := \{(p_1, ..., p_n; q_1, ..., q_m) | p_i \in \mathcal{H}, p_{i_1} \neq p_{i_2} \text{ for } i_1 \neq i_2 \text{ and } q_{j_1} \neq q_{j_2} \text{ for } j_1 \neq j_2\}.$$ 

and for $n \geq 2$ we define

$$Conf_n := \{(p_1, ..., p_n) | p_i \in \mathbb{C}, p_i \neq p_j \text{ for } i \neq j\}.$$ 

The groups of affine actions $G^{(1)} := \{z \mapsto az + b | a \in \mathbb{R}, b \in \mathbb{R}, a > 0\}$ and $G^{(2)} := \{z \mapsto az + c | a \in \mathbb{R}, c \in \mathbb{C}, a > 0\}$ act on $Conf_{n,m}$ and on $Conf_n$, respectively.

So we can define the configuration spaces

$$C_{n,m} := Conf_{n,m}/G^{(1)}$$

and

$$C_n := Conf_n/G^{(2)}.$$ 

By the conditions $n \geq 2$ and $2n + m \geq 2$ we avoid some nasty special cases. In particular, we can derive formulas for the dimensions of $C_{n,m}$ and $C_n$:

$$\dim(C_{n,m}) = 2n + m - 2, \quad \dim(C_n) = 2n - 3.$$ 

The permutation group $\Sigma_n \times \Sigma_m$ acts on $C_{n,m}$. We denote its symmetrization by $C_{A,B}$, where $A$ and $B$ are index sets with $n$ and $m$ elements, respectively. If we don’t have got index sets $A$ and $B$, we will write the same set as $C^{+}_{n,m}$.

5.2.2 Compactification of the configuration spaces

We first embed the space $C_{n,m}$ into a torus as follows:

Given an equivalence class $(p_1, ..., p_n; q_1, ..., q_m) \in C_{n,m}$, we can define $2n(n-1) + nm$ angles in $\mathbb{R}/2\pi\mathbb{Z}$ by taking

$$\text{Arg}(p_{i_1} - p_{i_2}), \quad i_1, i_2 = 1, ..., n \text{ and } i_1 \neq i_2$$

$$\text{Arg}(p_{i_1} - q_{i_2}), \quad i_1, i_2 = 1, ..., n \text{ and } i_1 \neq i_2$$

$$\text{Arg}(p_i - q_j), \quad i = 1, ..., n \text{ and } j = 1, ..., m.$$
In this way we get an embedding \( C_{n,m} \hookrightarrow (\mathbb{R}/2\mathbb{Z})^{2n(n-1)+nm} \). (It is easy to check that this map is well-defined and injective.)

The compactification \( \overline{C}_{n,m} \) of the configuration space is now simply defined as the closure of the image of \( C_{n,m} \) under this embedding.

Analogously, the compactification \( \overline{C}_n \) is obtained using only the angles \( \text{Arg}(p_i - p_{i+1}) \).

### 5.2.3 Boundary strata

It is not difficult to derive the boundary strata of the compactified configuration spaces. However, it is rather lengthy, so we will only give a very rough idea how the strata looks like and then quote the results we need. Precise calculations can be found in [FM].

A point \((p_1, ..., p_n; q_1, ..., q_m) \in \mathbb{C}^n \times \mathbb{R}^m \) belongs to \( \text{Conf}_{n,m} \), iff

- The \( p_i \) are pairwise distinct,
- the \( q_i \) are pairwise distinct and
- \( \Im(p_i) > 0 \).

So a point \((p_1, ..., p_n; q_1, ..., q_m) \in \mathbb{C}^n \times \mathbb{R}^m \) belongs to the boundary of \( \text{Conf}_{n,m} \), if one or several of the following conditions are met:

- Some of the \( p_i \) are equal,
- some of the \( q_i \) are equal and/or
- \( \Im(p_i) = 0 \) for some \( p_i \).

Since the group \( G^{(1)} \) respects these boundary properties, the boundary of \( C_{n,m} \) composes in the same way.

Following this path carefully yields a list of the boundary strata of \( C_{A,B} \) of codimension 1:

1. A boundary stratum of first type is characterized by a set \( S \subseteq A \), \( \#S \geq 2 \). The points of \( S \) move close to each other but stay away from \( \mathbb{R} \). We denote this stratum by \( \partial_S C_{A,B} \). It can be written as
   \[ \partial_S C_{A,B} \cong C_S \times C_{(A \setminus S) \cup \{\text{point}\}, B} \]

2. A boundary stratum of second type is characterized by two sets \( S \subseteq A \) and \( S' \subseteq B \), where \( 2\#S + \#S' \geq 2 \) and at least one point is left outside (i.e. \( \#S + \#S' \leq \#A + \#B \)). All these points move close to each other and to \( \mathbb{R} \). We denote this stratum by \( \partial_{S,S'} C_{A,B} \). It can be written as
   \[ \partial_{S,S'} C_{A,B} \cong C_{S,S'} \times C_{(A \setminus S), (B \setminus S') \cup \{\text{point}\}} \]
5.3 Admissible graphs

Definition 5.1 An admissible graph $\Gamma$ is an oriented graph such that:

1. there are two types of vertices; they are called vertices of first type and vertices of second type, respectively;

2. vertices of first type are labeled $1, \ldots, n$ and vertices of second type are labeled $1, \ldots, m$;

3. every edge starts at a vertex of first type;

4. there are no loops, i.e., no edges of the form $(v, v)$;

5. the number of edges is $2n + m - 2$.

For an admissible graph $\Gamma$, we will denote its set of vertices by $V_\Gamma$ and its set of edges by $E_\Gamma$. For every vertex $v$, let $In(v)$ be the set of all edges starting at $v$ and let $Out(v)$ be the set of all edges starting at $v$. (Note that $Out(v) = \emptyset$ if $v$ is of second type.)

Further, we define $G_{n,m}$ to be the set of all admissible graphs with $n$ vertices and $m$ edges.

6 Kontsevich’s proof

6.1 Construction of the $L_\infty$-isomorphism

6.1.1 Angle maps

On the configuration space $Conf_{2,0}$, we define a map $\phi' : Conf_{2,0} \to \mathbb{R}/2\pi\mathbb{Z}$ as follows: First, we endow the upper half plane $\mathcal{H}$ with the Lobachevsky metric. $\phi'(p, q)$ is the angle between the line from $p$ to $q$ and the vertical line through $p$. (Recall that in the Lobachevsky metric the line from $p$ to $q$ is a semicircle through $p$ and $q$ with midpoint on the real line.) This map is invariant under the group $G^{(1)}$, so it yields a map $C_{2,0} \to \mathbb{R}/2\pi\mathbb{Z}$. Finally, this map extends continuously to the angle map $\phi : \overline{C}_{2,0} \to \mathbb{R}/2\pi\mathbb{Z}$.

The next step links admissible graphs with our configuration spaces:

If $\Gamma \in G_{n,m}$ is an admissible graph, then every edge $e \in E_\Gamma$ defines a natural map (the forgetting map) $\overline{C}_{n,m} \to \overline{C}_{2,0}$ or $\overline{C}_{n,m} \to \overline{C}_{1,1} \subset \overline{C}_{2,0}$, depending on whether the target of the edge is of first type or of second type, respectively. In both cases, we end up with a map $\overline{C}_{n,m} \to \overline{C}_{2,0}$. (Of course, a priori there is only a forgetting map $Conf_{n,m} \to Conf_{2,0}$, but it is easy to check that this map extends as needed.) So we can define the angle map $\phi_e : \overline{C}_{n,m} \to \mathbb{R}/2\pi\mathbb{Z}$ as the pullback of the angle map of $\overline{C}_{2,0}$.
6.1 Construction of the $L_\infty$-isomorphism

6.1.2 The weights of a graph

Given an admissible graph $\Gamma$, we define its weight as follows:

$$W_\Gamma := \prod_{k=1}^{n} \frac{1}{(\#\text{Out}(k))!} \frac{1}{(2\pi)^{2n+m-2}} \int_{C_{n,m}^+} \bigwedge_{e \in E_\Gamma} d\phi_e.$$ 

Rigidly speaking, we cannot write down an expression like this without putting an orientation on the space $C_{n,m}^+$ and without ordering the set $E_\Gamma$. However, since we just wanted to give an impression of the means used in this proof, we ignore these technicalities. In fact, the choice of signs doesn’t have much influence on the results; as long as they are consequently used, the signs will take care of themselves. For a detailed examination see [AMM].

6.1.3 The pre-$L_\infty$-morphism associated to the graph

To an admissible graph $\Gamma$ with $n$ vertices, we will associate a linear map $U_\Gamma : \otimes^n T_{poly}(\mathbb{R}^d) \to D_{poly}(\mathbb{R}^d)[1 - n]$. We need to define it for homogeneous polyvector fields $\gamma$. $U_\Gamma(\gamma_1, ..., \gamma_n)$ is only non-zero, if the degree of $\gamma_i$ is $\#\text{Out}(i) - 1$ for all $i$. Recall that $\text{Out}(i)$ is the set of outgoing edges from the vertex $i$.

In order to write down a formula for $U_\Gamma$, we first define a function $\Phi_{v,I}$ for each vertex $v \in V_\Gamma$ and each labelling $I : E_\Gamma \to \{1, ..., d\}$ of the edges of $\Gamma$. We will use these functions to define $U_\Gamma(\gamma_1 \otimes ... \otimes \gamma_n)(f_1 \otimes ... \otimes f_n)$, and the $\Phi_{v,I}$ will implicitly depend on the $\gamma_i$ and the $f_i$. However, we don’t want to make this explicit because we want to avoid a flood of indices.

1. $v$ is a vertex of first type.

We have already seen that we may view a polyvector field $\gamma = \xi_1 \wedge ... \wedge \xi_k$ as a skew-symmetric tensor field $\sum_{\sigma \in \Sigma_k} \text{sgn}(\sigma) \xi_{\sigma(1)} \otimes ... \otimes \xi_{\sigma(k)}$.

Using this identification, we can define:

$$\gamma^{l_1,...,l_k} := \langle \gamma, dx^{l_1} \otimes ... \otimes dx^{l_k} \rangle.$$

Now we are ready to define $\Phi_{v,I}$:

$$\Phi_{v,I} := \left( \prod_{e \in I_n(v)} \partial_{I(e)} \right) \gamma^{l_1,...,l_k}_{v}.$$ 

where the $l_i$ are the labels of the outgoing edges from $v$.

2. $v$ is a vertex of second type In this case, we simply define

$$\Phi_{v,I} := \left( \prod_{e \in I_n(v)} \partial_{I(e)} \right) f_v,$$
Finally, we are ready to write down an expression for $U_{\Gamma}$:

$$U_{\Gamma}(\gamma_1 \otimes ... \otimes \gamma_n)(f_1 \otimes ... \otimes f_n) := \sum_{I:E_{\Gamma} \to \{1,...,d\}} \prod_{v \in V_{\Gamma}} \Phi_{v,I}.$$  

### 6.1.4 The quasi-isomorphism

Recall that in order to define an $L_\infty$-morphism $U : T_{poly}(\mathbb{R}^d) \to D_{poly}(\mathbb{R}^d)$, it is sufficient to define its $n$-th derivatives $U_n : \otimes^n T_{poly}(\mathbb{R}^d) \to D_{poly}(\mathbb{R}^d)[1-n]$:

$$U_n := \sum_{m \geq 0} \sum_{\Gamma \in G_{n,m}} W_{\Gamma} U_{\Gamma}.$$  

It is straightforward to check that $U$ is a pre-$L_\infty$-morphism. Furthermore, by a short consideration we see that the first Taylor-coefficient of $U$ is the map $U_1$ from section 5.1. Since this map induced a quasi-isomorphism of complexes, $U$ induces a quasi-isomorphism of complexes, too. The only remaining (and the difficult) part of the proof is showing that $U$ is also an $L_\infty$-morphism.

### 6.2 $U$ is an $L_\infty$-morphism

In order to prove that $U$ is an $L_\infty$-morphism, we need to show that $UQ^{(1)} = Q^{(2)}U$, where $Q^{(1)}$ and $Q^{(2)}$ are the codifferentials of the $L_\infty$-algebras $T_{poly}(\mathbb{R}^d)$ and $D_{poly}(\mathbb{R}^d)$, respectively.

Now recall that we can describe these codifferentials in terms of their Taylor coefficients. So we can simplify the condition to

$$(UQ^{(1)})_n = (Q^{(2)}U)_n \text{ for all } n \geq 0.$$  

After plugging in $Q^{(1)}$ and $Q^{(2)}$, and simplifying, we end up with:

$$\sum_{i \neq j} \pm(U_{n-1}((\gamma_i \bullet \gamma_j) \otimes \gamma_1 \otimes ... \otimes \gamma_n))(f_1 \otimes ... \otimes f_m) +$$

$$+ \sum_{k,l \geq 0, k+l=n} \frac{1}{k!l!} \sum_{\sigma \in \Sigma_n} \pm(U_k(\gamma_{\sigma(1)} \otimes ... \otimes \gamma_{\sigma(k)})) \circ (U_l(\gamma_{\sigma(k+1)} \otimes ... \otimes \gamma_{\sigma(n)}))(f_1 \otimes ... \otimes f_m) = 0,$$

where $\bullet$ and $\circ$ are the operations we used to define the brackets of $D_{poly}(X)$ and $T_{poly}(X)$, respectively (see definition 1.3 and 2.2).

Recall that $U$ was defined as $\sum_{\Gamma} W_{\Gamma} U_{\Gamma}$. If we use this in the formula above, we get a linear combination of expressions of the form

$$(U_{\Gamma}((\gamma_i \bullet \gamma_j) \otimes \gamma_1 \otimes ... \otimes \gamma_n))(f_1 \otimes ... \otimes f_m)$$  

(2)
6.2 $U$ is an $L_{\infty}$-morphism

and of the form

$$(U_\Gamma_1(\gamma_{\sigma(1)} \otimes \ldots \otimes \gamma_{\sigma(k)}) \circ (U_\Gamma_2(\gamma_{\sigma(k+1)} \otimes \ldots \otimes \gamma_{\sigma(n)}))(f_1 \otimes \ldots \otimes f_m). \quad (3)$$

We would like to rewrite our equation as a linear combination just of the

$$U_\Gamma(\gamma_1 \otimes \ldots \otimes \gamma_n)(f_1 \otimes \ldots \otimes f_n). \quad (4)$$

In order to do this, we must replace the expressions (2) and (3) by a linear combination of expressions of type (4).

6.2.1 Getting rid of the $\bullet$

We want to replace the expression (2). Recall the definition

$$U_\Gamma = \sum_{I: E_\Gamma \to \{1, \ldots, d\}} \prod_{v \in V_\Gamma} \Phi_{v,I}.$$  

The sum won’t interfere with our purpose to rewrite the expression as a certain linear combination. Furthermore, $\Phi_{v,I}$ will only depend on the first argument, if $v = 1$. (The vertices of fist type were labelled by $1, \ldots, n$, so this is just another way of saying that $v$ is the first vertex of first type). Hence, we just have to deal with $\Phi_{1,I}$.

The definition of $\Phi_{1,I}$ was

$$\Phi_{1,I}(\gamma_i \bullet \gamma_j) := \left( \prod_{e \in E_{\text{in}}(1)} \partial_{I(e)} \right) (\gamma_i \bullet \gamma_j)^{l_1, \ldots, l_k},$$

where the $l_i$ are the labels of the outgoing edges from $1$.

Using the definition of $\bullet$, we may replace $(\gamma_i \bullet \gamma_j)^{l_1, \ldots, l_k}$ by

$$\sum_{l=1}^{d} \sum_{Q \subseteq \text{out}(1)} \pm \gamma_{i,lQ} \gamma_{j,\text{out}(1)\setminus Q}.$$  

The notation $\gamma_{i,lQ}$ means simply $\gamma_{i,lq_1, \ldots, q_r}$, where $Q = \{q_1, \ldots, q_r\}$. (Recall that terms of the form $\gamma^Q$ are defined to be zero, if the number of elements in $Q$ does not match the degree of $\gamma$.) After replacing, we get

$$\Phi_{1,I}(\gamma_i \bullet \gamma_j) := \left( \prod_{e \in E_{\text{in}}(1)} \partial_{I(e)} \right) \left( \sum_{l=1}^{d} \sum_{Q \subseteq \text{out}(1)} \pm \gamma_{i,lQ} \gamma_{j,\text{out}(1)\setminus Q} \right).$$

Now we apply Leibniz rule and get

$$\sum_{l=1}^{d} \sum_{P \subseteq \text{in}(1)} \sum_{Q \subseteq \text{out}(1)} \pm \partial_{I(P)} \gamma_{i,lQ} \partial_{I(\text{in}(1)\setminus P)} \gamma_{j,\text{out}(1)\setminus Q}.$$  

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Let us briefly summarize what we have done so far in order to eliminate the bullet in our original formula (2). We have shown that (2) can be written as a linear combination of terms of the form $\partial_I(P)\gamma_i^{l_1}Q\partial_I(\text{in}(1)\setminus P)\gamma_j^{\text{out}(1)\setminus Q}$. We want to get a linear combination only of terms like $U_{\Gamma'}(\gamma_1 \otimes \ldots \otimes \gamma_n)$. So what is left to do is constructing a graph $\Gamma'$ such that

$$U_{\Gamma'}(\gamma_1 \otimes \ldots \otimes \gamma_n) = \partial_I(P)\gamma_i^{l_1}Q\partial_I(\text{in}(1)\setminus P)\gamma_j^{\text{out}(1)\setminus Q}.$$  

Note that $\text{in}(1)$ and $\text{out}(1)$ on the right hand side are the in- and outgoing edges of the vertex 1 of the original graph, which we will call $\Gamma$. We will of course construct $\Gamma'$ out of $\Gamma$:

$\Gamma'$ is obtained from $\Gamma$ by splitting the vertex 1 into two new vertices $i$ and $j$. The other vertices are renamed, so that they keep their order. The incoming and outgoing edges of $i$ are $P$ and $Q$, respectively, and the incoming and outgoing edges of $j$ are $\text{in}(1)\setminus P$ and $\text{out}(1)\setminus Q$, respectively. Finally, there is one additional edge going from $i$ to $j$, labelled by $l$. We will give an example for this process:

![Graph](image)

These graphs are obtained from $\Gamma$ by splitting the vertex 1 into two new vertices 1 and 3. (Note that the vertex 3 in $\Gamma$ is renamed in $\Gamma'$.) The edges are distributed between the new vertices via the sets $P$ and $Q$.

It is not too hard to see that this construction gives an admissible graph, which has exactly the required $U_{\Gamma'}$. 

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6.2.2 Getting rid of the ◦

We want to replace expression (3):

\[ U_{Γ_1}(γ_{σ(1)} ⊗ ... ⊗ γ_{σ(k)}) ◦ U_{Γ_2}(γ_{σ(k+1)} ⊗ ... ⊗ γ_{σ(n)}) \]

The procedure will be very similar to the way we proceeded in the paragraph above.

First of all, we recall the definition of ◦:

\[
(φ_1 ◦ φ_2)(a_1 ⊗ ... ⊗ a_{k_1+k_2}) := \sum_{i=1}^{k_1} (-1)^{(i-1)k_2} φ_1(a_1 ⊗ ... ⊗ a_{i-1} ⊗ (φ_2(a_i ⊗ ... ⊗ a_{i+k_2}) ⊗ a_{i+k_2+1} ⊗ ... ⊗ a_{k_1+k_2})
\]

After using the definitions of \(U_{Γ}\) and ◦, and after simplifying the result, we finally get a sum over terms of the form:

\[
±W_{Γ_1} W_{Γ_2} \prod_{v_1 ∈ V_{Γ_1}\setminus \tilde{Γ}} φ_{v_1} \prod_{v_2 ∈ V_{Γ_2}} η_{v_2} \phi_{v_2}.
\]

In this term, \(i\) is the index from the definition of ◦, and \(P_{v_2}\) is a partition of \(in(\tilde{Γ})\), indexed by \(V_{Γ_2}\), i.e.

\[
\bigcup_{v_2 ∈ V_{Γ_2}} P_{v_2} = in(\tilde{Γ}).
\]

This partition occurs because (as in the case of •) we made use of Leibniz rule. This rule tells us to sum over all possible ways to associate the derivations to the factors of the product. A convenient way to put this into a formula is via partitions.

So we need to construct a graph \(Γ'\) such that

\[
U_{Γ'}(γ_1 ⊗ ... ⊗ γ_n) = \prod_{v_1 ∈ V_{Γ_1}\setminus \tilde{Γ}} φ_{v_1} \prod_{v_2 ∈ V_{Γ_2}} η_{v_2} \phi_{v_2}.
\]

\(Γ'\) is obtained from \(Γ_1\) by removing the vertex \(\tilde{Γ}\) and replacing it by the graph \(Γ_2\). The edges ending formerly at \(\tilde{Γ}\) end now at a vertex of \(Γ_2\) and the partition \(P\) tells us exactly which edge ends at which vertex.

Again, it is easy to check that \(Γ'\) is an admissible graph with the required property.
Here is an example to illustrate this process:

\[
\begin{array}{c}
\text{the graph } \Gamma_1 \\
\end{array}
\]

\[
\begin{array}{c}
\text{the graph } \Gamma_2 \\
\end{array}
\]

In \( \Gamma_1 \), the vertex 3 is replaced by the graph \( \Gamma_2 \).

Finally, the nodes are renamed and the edges that ended at 3 end now at some vertices of \( \Gamma_2 \).

### 6.2.3 Vanishing of the coefficients

We have now shown that we can write the condition for \( U \) to be an \( L_\infty \)-morphism in the form

\[
\sum_{\Gamma} c_\Gamma U_\Gamma(\gamma_1 \otimes ... \otimes \gamma_n) = 0,
\]

where \( c_\Gamma \) are certain coefficients.

We will show the equation by proving (exemplary) that \( c_\Gamma = 0 \) for all admissible graphs \( \Gamma \).
6.2 \( U \) is an \( L_\infty \)-morphism

In order to do this, we will examine integrals

\[
\int_{\partial \mathcal{C}_{n,m}} \bigwedge_{e \in E_T} d\phi_e = \int_{\mathcal{C}_{n,m}} d \left( \bigwedge_{e \in E_T} d\phi_e \right) = 0,
\]

and show that these are the coefficients \( c_T \) of the graph \( \Gamma \).

In section (3.1.3) we listed two types of boundary strata. Kontsevich distinguishes several subcases for each type. We will only treat one case properly and simply quote the result for the other cases.

Let \( \Gamma \) be an admissible graph.

Assume we are given two sets \( S \subseteq \{1, \ldots, n\} \) and \( S' \subseteq \{\overline{1}, \ldots, \overline{m}\} \). These two sets characterize a boundary stratum \( F \) of second type (The points from \( S \) and \( S' \) move close to each other, see section (3.1.3)). We assume further that there is no "bad edge", i.e. an edge \((i, j)\) such that \( i \in S \) and \( j \in \{1, \ldots, n\} \setminus S \).

Recall how we constructed a graph \( \Gamma' \) out of two graphs \( \Gamma_1 \) and \( \Gamma_2 \) in the previous subsection. Let us define \( \Gamma_1 \) as the graph obtained from \( \Gamma \) by collapsing the sets \( S \) and \( S' \) to a vertex of second type and let \( \Gamma_2 \) be the subgraph of \( \Gamma \) with vertices \( S \cup S' \) (and all possible edges). Then the graph \( \Gamma' \) from section (3.4.2) is exactly our original \( \Gamma \).

On the other hand, abbreviating \( \mathcal{C}_{S;S'} \) by \( C_1 \) and \( \mathcal{C}_{\{1, \ldots, n\} \setminus S; \{\overline{1}, \ldots, \overline{m}\} \setminus S' \cup \text{point} } \) by \( C_2 \), we have

\[
\int_F \bigwedge_{e \in E_T} d\phi_e = \pm \int_{C_1 \times C_2} \bigwedge_{e \in E_T} d\phi_e =
\]

\[
= \pm \int_{C_1 \times C_2} \bigwedge_{e \in E_{T_1}} d\phi_e \wedge \bigwedge_{e \in E_{T_2}} d\phi_e =
\]

\[
= \pm \int_{C_1} \bigwedge_{e \in E_{T_1}} d\phi_e \int_{C_2} \bigwedge_{e \in E_{T_2}} d\phi_e =
\]

\[
= \pm W_{T_1} W_{T_2}.
\]

But this term is in fact the coefficient of \( U_T \), as we computed earlier.

Going through the other cases in the same way, one gets the following list:

- boundary stratum of second type, no bad edges: This corresponds to the contributions of \( \circ \). (This is the case we elaborated above.)

- boundary stratum of second type, with bad edges: In this case, the integral over \( F \) is zero.
7 FURTHER DEVELOPMENTS

- boundary stratum of first type, exactly two points approach each other: This corresponds to the contributions of .
- boundary stratum of first type, more than two points approach each other: The integral over $F$ is zero.

Hence, we see that the coefficients $c_f$ match the integrals as claimed. Therefore, they vanish, and $U$ is an $L_\infty$-morphism.

7 Further developments

Let us explain why this proof attracted so much attention.

First of all, it was the first solution of the deformation quantization problem, which had been unsolved for several years. Moreover, Kontsevich proved much more than just this problem: He proved the formality theorem, which establishes an important categorial link.

The formality theorem also provides a new access to some other problems: For example, by using the theorem one can compute an explicit formula for the $\ast$-products in some important quantum spaces like the $q$-deformed Minkowski space and the $q$-deformed Euclidean space, see [WW] and [Man].

As it is with most theorems, there are various ways to generalize the formality theorem. In [TT], D. Tamarkin and B. L. Tsygan formulated several formality conjectures for Hochschild cochains. (They also prove some theorems which seem to indicate that their conjectures may be correct).

The usage of new means in the proof, however, might be even more important than the formality theorem itself. Let us briefly repeat what we used in order to define $U$: Basically, we summed over a set of graphs. For each graph, we had a certain weight, which is defined as an integral over a configuration spaces. Exactly the same method was used in quantum mechanics by Richard Feynman. His invention of the so-called "Feynman diagrams" is considered as a breakthrough and was one of the reasons for his Nobel prize in 1965. When Kontsevich used the same tools in mathematical quantization theory, it aroused hope that this might become a similar success story.

And indeed, there have been some encouraging developments in the past few years. A. Cattaneo and G. Felder gave in 2000 a new, more conceptual definition for the formal deformation of a Poisson manifold via a so-called Feynman integral, see [CF].

Finally, surprisingly many links between the formality theorem and other parts of mathematics have been established since 1997. V. Kathotia discovered some remarkable links between deformation quantization and the Baker-Campbell-Hausdorff (BCH) formula. In particular, he shows that it is possible to derive the BCH-formula from Kontsevich's formula for deformation quantization. By this, he can derive certain properties of Kontsevich's quasi-isomorphism (e.g. a corollary is that the weights of non-loop-graphs are
rational). Furthermore, he conjectures that by the BCH-formula, it might be possible to find a universal formula for deformation quantization of $\mathbb{R}^d$. However, the most surprising piece of work is due to D. Tamarkin, who gave a completely different proof to the formality theorem by proving a very general theorem about operads, cf. [Tam]. Several authors developed these ideas further, e.g. [Vor] and [Vo2]; Kontsevich gives a good overview of the situation in [Ko2].

References

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