A Short Proof of the Random Ramsey Theorem

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Abstract. In this paper we give a short proof of the Random Ramsey Theorem of Rödl and Ruciński: for any graph $F$ which contains a cycle and $r \geq 2$, there exist constants $c, C > 0$ such that

$$P[G_{n,p} \rightarrow (F)^e_r] = \begin{cases} 1 - o(1), & p \geq C_n^{-1/m_2(F)} \\ o(1), & p \leq cn^{-1/m_2(F)} \end{cases},$$

where $m_2(F) = \max_{J \subseteq F, v_J \geq 3} d_2(J)$.

The proof of the 1-statement is based on the recent beautiful hypergraph container theorems by Saxton/Thomason and Balogh/Morris/Samotij. The proof of the 0-statement is elementary.

1. Introduction

For graphs $G$ and $F$ and a constant $r \in \mathbb{N}$, we denote with $G \rightarrow (F)^e_r$ the property that every edge-colouring of $G$ with $r$ colours (we call this an $r$-colouring) contains a copy of $F$ with all edges having the same colour. Ramsey’s theorem then implies that for all graphs $F$ and $r$ we have $K_n \rightarrow (F)^e_r$, for $n$ large enough. At first sight it is not immediately clear whether this follows from the density of $K_n$ or its rich structure. As it turns out, studying Ramsey properties of random graphs shows that the latter is the case, as random graphs give examples of sparse graphs with the desired Ramsey property.

The study of Random Ramsey Theory was initiated by Luczak, Ruciński, and Voigt [7] who studied the Ramsey property of random graphs in the vertex-colouring case and also established the threshold for the property $G_{n,p} \rightarrow (K_3)^2$. Thereupon, in a series of papers Rödl and Ruciński [9, 10, 11] determined the threshold of $G_{n,p} \rightarrow (F)^e_r$, in full generality. Formally, their result reads as follows.

For every graph $G$ we denote by $V(G)$ and $E(G)$ its vertex and edge sets and by $v_G$ and $e_G$ their sizes. For every graph $G$ on at least 3 vertices we set $d_2(G) = (e_G - 1)/(v_G - 2)$. By $m_2(G)$ we denote for every graph $G$ the so-called 2-density, defined as

$$m_2(G) = \max_{J \subseteq G, v_J \geq 3} d_2(J).$$

If $m_2(G) = d_2(G)$ then we say that a graph $G$ is 2-balanced, and if in addition $m_2(G) > d_2(J)$ for every subgraph $J \subset G$ with $v_J \geq 3$, we say that $G$ is strictly 2-balanced.

Theorem 1 (Rödl, Ruciński [9, 10, 11]). Let $r \geq 2$ and let $F$ be a fixed graph that is not a forest of stars or, in the case $r = 2$, paths of length 3. Then there exist positive constants $c = c(F, r)$, and $C = C(F, r)$ such that

$$\lim_{n \to \infty} P[G_{n,p} \rightarrow (F)^e_r] = \begin{cases} 0 & \text{if } p \leq cn^{-1/m_2(F)} \\ 1 & \text{if } p \geq Cn^{-1/m_2(F)} \end{cases}.$$
For the exceptional case of a star with $k$ edges it is easily seen that the threshold is determined by the appearance of a star with $r(k – 1) + 1$ edges. For the path $P_3$ of length three the 0-statement only holds for $p \ll n^{-1/m_2(P_3)} = n^{-1}$ since, for example, a $C_3$ with a pendant edge at every vertex has density one and cannot be 2-coloured without a monochromatic $P_3$.

Note that $p = n^{-1/m_2(F)}$ is the density where we expect that every edge is contained in roughly a constant number of copies of $F$. This observation can be used to provide an intuitive understanding of the bounds of Theorem 1. If $c$ is very small, then the number of copies of $F$ is w.h.p. (with high probability, i.e., with probability $1 – o(1)$ if $n$ tends to infinity) small enough that they are so scattered that a colouring without a monochromatic copy of $F$ can be found. If, on the other hand, $C$ is big then these copies w.h.p. overlap so heavily that every colouring has to induce at least one monochromatic copy of $F$.

The aim of this paper is to give a short proof of Theorem 1.

## 2. Proof of the 1-statement

The proof of the 1-statement requires two tools. The first one is a well-known quantitative strengthening of Ramsey’s theorem. We include its short proof for convenience of the reader.

**Theorem 2** (Folklore). For every graph $F$ and every constant $r \geq 2$ there exist constants $\alpha > 0$ and $n_0$ such that for all $n \geq n_0$, every $r$-colouring of the edges of $K_n$ contains at least $\alpha n^{r-1}$ monochromatic copies of $F$.

**Proof.** From Ramsey’s theorem we know that there exists $N := N(F, r)$ such that every $r$-colouring of the edges of $K_N$ contains a monochromatic copy of $F$. Thus, in any $r$-colouring of $K_n$ every $N$-subset of the vertices contains at least one monochromatic copy of $F$. As every copy of $F$ is contained in at most $\binom{n-2}{N-2}$ $N$-subsets, the theorem follows e.g. with $\alpha = \frac{1}{N^{r-1}}$. \qed

We will need in particular the following easy consequence of Theorem 2

**Corollary 3.** For every graph $F$ and every $r \in \mathbb{N}$ there exist constants $n_0$ and $\delta, \varepsilon > 0$ such that the following is true for all $n \geq n_0$. For any $E_1, \ldots, E_r \subseteq E(K_n)$ such that for all $1 \leq i \leq r$ the set $E_i$ contains at most $\varepsilon n^{r-1}$ copies of $F$, we have

$$|E(K_n) \setminus (E_1 \cup \ldots \cup E_r)| \geq \delta n^2.$$

**Proof.** Let $\alpha$ and $n_0$ be as given by Theorem 2 for $F$ and $r + 1$, and set $\varepsilon = \alpha / 2r$. Further, let $E_{r+1} := E(K_n) \setminus (E_1 \cup \ldots \cup E_r)$ and consider the colouring $\Delta : E(K_n) \to [r + 1]$ given by $\Delta(e) = \min \{i \in [r + 1] : e \in E_i\}$. By Theorem 2 there exist at least $\alpha n^{r-1}$ monochromatic copies of $F$, of which, by assumption on the sets $E_i$, at least $\frac{1}{n^{r-1}} \cdot \alpha \cdot n^{r-1}$ must be contained in $E_{r+1}$. As every edge is contained in at most $2n^{r-2}$ copies of $F$ the claim of the corollary follows e.g. for $\delta = \frac{2n^{r-2}}{\alpha n^{r-1}}$. \qed

The second tool that we need is a consequence of the beautiful container theorems of Balogh, Morris and Samotij [11] and Saxton and Thomason [12]. The following theorem is from Saxton and Thomason, who obtain it for all graphs $F$. Balogh, Morris and Samotij obtain a similar statement for all 2-balanced graphs $F$.

**Definition 4.** For a given set $S$ and constants $k \in \mathbb{N}, s > 0$, let $T_{k,s}(S)$ be the family of $k$-tuples of subsets defined as follows,

$$T_{k,s}(S) = \{(S_1, \ldots, S_k) \mid S_i \subseteq S \text{ for } 1 \leq i \leq k \text{ and } \left| \bigcup_{i=1}^{k} S_i \right| \leq s\}.$$
Theorem 5 ([12], Theorem 1.3). For any graph $F$ and $\varepsilon > 0$, there exist $n_0$ and $k > 0$ such that the following is true. For every $n \geq n_0$ there exist $t = t(n)$, pairwise distinct tuples $T_1, \ldots, T_t \in T_{k, kn^{-1/m_2(F)}}(E(K_n))$ and sets $C_1, \ldots, C_t \subseteq E(K_n)$, such that

(a) each $C_i$ contains at most $\varepsilon n^{v_F}$ copies of $F$,
(b) for every $F$-free graph $G$ on $n$ vertices there exists $1 \leq i \leq t$ such that $T_i \subseteq E(G) \subseteq C_i$. (Here $T_i \subseteq E(G)$ means that all sets contained in $T_i$ are subsets of $E(G)$.)

Note that the main result in [12] is more general, as it provides a similar structure for independent sets in uniform hypergraphs.

With these two tools in hand the proof of the 1-statement of Theorem 1 is now easily completed.

Proof of Theorem 1 (1-statement). Let $\varepsilon$ and $\delta$ be as in Corollary 3 and let $n_0$ and $k$ be as in Theorem 5 for $F$ and $\varepsilon$, and assume that $n \geq n_0$. If $G_{n,p} \not\to (F)_{\varepsilon}^r$, then there exists a colouring $\Delta : E(G_{n,p}) \to r$ such that for all $1 \leq j \leq r$ the set $E_j := \Delta^{-1}(j)$ does not contain a copy of $F$. By Theorem 5 we have that for every such $E_j$ there exists $1 \leq i_j \leq t(n)$ such that $T_{i_j} \subseteq E_j \subseteq C_{i_j}$ and $C_{i_j}$ contains at most $\varepsilon n^{v_F}$ copies of $F$. The trivial, but nonetheless crucial observation is that $G_{n,p}$ completely avoids $E(K_n) \setminus (C_{i_1} \cup \ldots \cup C_{i_r})$, which by Corollary 3 has size at least $\delta n^2$.

Therefore we can bound $P[G_{n,p} \not\to (F)_{\varepsilon}^r]$ by bounding the probability that there exist tuples $T_{i_1}, \ldots, T_{i_r}$ that are contained in $G_{n,p}$ such that $E_0(T_{i_1}, \ldots, T_{i_r}) := E(K_n) \setminus (C_{i_1} \cup \ldots \cup C_{i_r})$ is edge-disjoint from $G_{n,p}$. Thus

$$P[G_{n,p} \not\to (F)_{\varepsilon}^r] \leq \sum_{i_1, \ldots, i_r} P[T_{i_1}, \ldots, T_{i_r} \subseteq G_{n,p} \cap G_{n,p} \cap E_0(T_{i_1}, \ldots, T_{i_r}) = \emptyset],$$

where $i_1, \ldots, i_r$ run over the choices given by Theorem 5. Note that the two events in the above probability are independent and the probability can thus be bounded by

$$p^{\bigcup_{j=1}^r T_{i_j}^+} \cdot (1 - p)^{\delta n^2},$$

where by $T_{i_j}^+$ we denote the union of the sets of the $k$-tuple $T_{i_j}$. The sum can be bounded by first deciding on $s := |\bigcup_{j=1}^r T_{i_j}^+| \leq r \cdot kn^{-1/m_2(F)}$, then choosing that many edges $\left(\binom{n}{2}ight)$ choices) and finally deciding for every edge in which sets of the $k$-tuples $T_{i_j}$ it appears $\left(2^s\right)$ choices. Together, this gives

$$P[G_{n,p} \not\to (F)_{\varepsilon}^r] \leq (1 - p)^{\delta n^2} \cdot \sum_{s=0}^{rkn^{-1/m_2(F)}} \binom{n}{s} (2^s)^s p^s \leq e^{-\delta n^2 p} \cdot \sum_{s=0}^{rkn^{-1/m_2(F)}} \left(\frac{e2^k n^2 p}{2s}\right)^s.$$

Recall that $p = Cn^{-1/m_2(F)}$. By choosing $C$ sufficiently large (with respect to $k$) we may assume that

$$\sum_{s=0}^{rkn^{-1/m_2(F)}} \left(\frac{e2^k n^2 p}{2s}\right)^s \leq n^2 \cdot \left(\frac{e2^k C}{2r k}\right)^{rkn^{-1/m_2(F)}p} \leq e^{\frac{1}{2} \delta n^2 p}$$

and thus $P[G_{n,p} \not\to (F)_{\varepsilon}^r] = o(1)$, as desired. \qed
We remark that the same approach, with Theorem 2 and Theorem 5 replaced with the corresponding hypergraph versions, gives an alternative proof of the 1-statement for a Random Ramsey Theorem for hypergraphs obtained by Friedgut, Rödl and Schacht [5] and Conlon and Gowers [3].

3. Proof of the 0-statement

We need to show that with high probability the edges of a random graph $G_{n,p}$ with $p = c n^{-1/m_2(F)}$, for $0 < c = c(F) < 1$ small enough, can be coloured in such a way that we have no monochromatic copy of $F$. If $m_2(F) = 1$ we have $p \leq cn^{-1}$ with $c < 1$. It is well-known that then every component of $G_{n,p}$ is w.h.p. either a tree or a unicyclic graph (see [4]). One easily checks that we can colour each such component without a monochromatic copy of $F$ whenever $F$ is not a star and not a path of length 3 (or $r \geq 3$ in the latter). In the following we thus assume that $m_2(F) > 1$.

Observe that we may also assume without loss of generality that $r = 2$ and that $F$ is strictly 2-balanced. If not, replace $F$ by a minimal subgraph $F'$ with the same 2-density. Clearly, if we find a 2-colouring of the edges of $G_{n,p}$ without a monochromatic copy of $F'$ this 2-colouring will also contain no monochromatic copy of $F$.

The expected number of copies of $F$ on any given edge is bounded by

$$2e_F \cdot n^{v_F - 2} \cdot p^{e_F - 1} \leq 2e_F \cdot c^{e_F - 1}.$$  

That is, for $0 < c < 1$ small enough we do not expect more than one copy. We now show why this makes the colouring process easier.

Let $e$ be an edge in $G_{n,p}$. Assume that $G_{n,p} - e$ is 2-colourable without a monochromatic copy of $F$. Consider any such colouring. If this colouring cannot be extended to $e$ then there has to exist both a red and a blue copy of $F - \hat{e}$ (for some $\hat{e} \in E(F)$ which might be different in these two copies) such that $e$ completes both of these copies to a copy of $F$. Since these two copies of $F - \hat{e}$ are in different colours, and thus edge disjoint, we can conclude that there exist at least two copies of $F$ which intersect in $e$. In other words: a necessary obstruction for extending a colouring from $G - e$ to $G$ is that $e$ is contained in at least two copies of $F$ that only intersect in $e$.

To formalize this idea, call an edge $e$ closed in $G$ if it is contained in at least two copies of $F$ whose edge sets intersect exactly in $e$. (Note: we do allow that the vertex sets of these copies intersect in more than two vertices). Otherwise we call the edge open. With this notion at hand we can now formulate the following algorithm for obtaining the desired 2-coloring of $G_{n,p}$:

\[
\hat{G} := G_{n,p}; \\
\textbf{while} \text{ there exists an open edge } e \text{ in } \hat{G} \text{ do} \\
\hat{G} \leftarrow \hat{G} - e; \\
\text{colour } \hat{G}; \\
\text{add the edges in reverse order and colour them appropriately.}
\]

The critical point, of course, is the statement 'colour $\hat{G}$'. We need to show that this step is indeed possible.

Observe that after termination of the while-loop the graph $\hat{G}$ has the following property: every edge of $\hat{G}$ is closed. It is easy to see that $\hat{G}$ is actually the (unique) maximal subgraph of $G_{n,p}$ with the property that every edge is closed (within this subgraph). We call $\hat{G}$ the $F$-core of $G_{n,p}$.

We now further refine $\hat{G}$. Consider an auxiliary graph $G_F$ defined as follows: the set of vertices correspond to the set of copies of $F$ in $\hat{G}$ and two vertices
are connected by an edge if and only if the corresponding copies of $F$ have at least one edge in common. Since every edge of $\hat{G}$ belongs to a copy of $F$, the connected components of $G_F$ naturally partition the edges of $\hat{G}$ into equivalence classes. Observe that, by definition, each equivalence class (an $F$-component for short) can be coloured separately in order to find a valid colouring of the $F$-core. Note also that within $\hat{G}$ the $F$-components need not necessarily form components, cf. Figure 1.

The core of our argument is the following lemma, which states that with high probability every $F$-component in the $F$-core of $G_{n,p}$ has constant size.

**Lemma 6.** Let $F$ be a strictly $2$-balanced graph with $e_F \geq 3$. There exist $c = c(F) > 0$ and $L = L(F) > 0$ such that if $p \leq cn^{-1/m_2(F)}$ then w.h.p. every $F$-component of the $F$-core of $G_{n,p}$ has size at most $L$.

We will prove Lemma 6 in the next subsection. Before doing so we show how it can be used to complete the proof of Theorem 1. For that we also need the following result of Rödl and Ruciński [9], which states that graphs with small enough density do not have the Ramsey property. We include its short proof in the appendix of this paper.

**Theorem 7 ([9]).** Let $G$ and $F$ be two graphs. If $m(G) \leq m_2(F)$, where $m(G) = \max_{J \subseteq G} \frac{e_J}{v_J}$ is the maximal density of $G$, and $m_2(F) > 1$ then $G \not\rightarrow (F)^2_2$.

With these results in hand, the proof of the $0$-statement of Theorem 1 is straightforward.

**Proof of Theorem 1 (0-statement).** Recall that we may assume w.l.o.g. that $F$ is strictly $2$-balanced and that $m_3(F) > 1$. Choose $c = c(F)$ and $L = L(F)$ according to Lemma 6. Then $G_{n,p}$ has w.h.p. the property that every $F$-component of the $F$-core of $G_{n,p}$ has size at most $L$.

Observe that there exist only constantly many different graphs on at most $L$ vertices. Let $G$ be one such graph, and choose $G' \subseteq G$ such that $m(G) = e_{G'}/v_{G'}$. Then the expected number of copies of $G'$ in $G_{n,p}$ is bounded by $n^{e_{G'}/v_{G'}}$. Observe that for $p = cn^{-1/m_2(F)}$ we have $n^{e_{G'}/v_{G'}} = o(1)$ whenever $m(G) = e_{G'}/v_{G'} > m_2(F)$. It follows from Markov’s inequality that for $p \leq cn^{-1/m_2(F)}$ w.h.p. there is no copy of $G'$, and hence no copy of $G$ in $G_{n,p}$. Therefore, w.h.p. every subgraph $G$ of $G_{n,p}$ of size $|V(G)| \leq L$ satisfies $m(G) \leq m_2(F)$.

Combining both properties we get: with high probability all $F$-components $G$ of $G_{n,p}$ satisfy $m(G) \leq m_2(F)$ and Theorem 7 thus implies that there exists a
2-colouring of $G$ without a monochromatic copy of $F$. The union of these edge colourings of all $F$-components therefore yields the desired colouring of the $F$-core of $G_{n,p}$. Finally, as explained above this colouring can be extended to a colouring of $G_{n,p}$ without a monochromatic copy of $F$. \hfill \Box

3.1. **Proof of Lemma 6** We start by collecting some properties of strictly 2-balanced graphs.

**Lemma 8.** If $F$ is strictly 2-balanced, then $F$ is 2-connected.

**Proof.** Clearly, $F$ is connected. As then $(e_F - 2)/(v_F - 3) \geq (e_F - 1)/(v_F - 2)$, we deduce that $F$ cannot contain a vertex of degree 1. Assume there exists $v \in V(F)$ that is a cut vertex. Then there exist subgraphs $F_1$ and $F_2$ that both contain at least three vertices such that $F_1 \cup F_2 = F$ and $V(F_1) \cap V(F_2) = \{v\}$. As $F$ is strictly 2-balanced we get (using $a/b < x$ implies $(a + c)/(b + d) < x$) $e_F - 2 = (e_{F_1} - 1) + (e_{F_2} - 1) < m_2(F) \cdot (v_{F_1} - 2 + v_{F_2} - 2) = m_2(F) \cdot (v_F - 3)$. Since $m_2(F) = (e_F - 1)/(v_F - 2)$ (as $F$ is balanced), this contradicts the inequality from above. \hfill \Box

**Lemma 9.** Let $F$ be strictly 2-balanced and let $G$ be an arbitrary graph. Construct a graph $\hat{G}$ by attaching $F$ to an edge $e$ of $G$. Then $\hat{G}$ has the property that if $\hat{F}$ is a copy of $F$ in $\hat{G}$ that contains a least one vertex from $F - e$, then $\hat{F} = F$.

**Proof.** Assuming the opposite, let $\hat{F}$ be a copy of $F$ which violates the claim and set $F_g = \hat{F}[V(G)]$ and $F_f = \hat{F}[V(F)]$. The fact that $\hat{F}$ violates the claim implies that $\hat{F}$ contains at least one vertex from $F - e$ and at least one from $G - e$. Since every strictly 2-balanced graph is, by Lemma 8 2-connected, it follows that both vertices of $e$ belong to $V(\hat{F})$, thus $v_F = v_{F_g} + v_{F_f} - 2$ and $v_{F_g}, v_{F_f} \geq 3$.

![Figure 2](image.png)

**Figure 2.** Situation of Lemma 9: copy $F$ is attached to $G$ at edge $e$; copy $\hat{F}$ intersects the interior of both $F$ and $G$.

If $e \notin \hat{F}$, we add edge $e$ to $F_f$. Then $e_F = e_{F_g} + e_{F_f} - 1$ regardless of whether $e \in \hat{F}$ or not. As $F_g$ and $F_f$ are strict subgraphs of $F$, we have

$$
\frac{e_{F_g} - 1}{v_{F_g} - 2} < m_2(F) \quad \text{and} \quad \frac{e_{F_f} - 1}{v_{F_f} - 2} < m_2(F)
$$

since $F$ is strictly 2-balanced. This, however, yields a contradiction, as

$$
m_2(F) = \frac{e_F - 1}{v_F - 2} = \frac{e_{F_g} - 1 + e_{F_f} - 1}{v_{F_g} - 2 + v_{F_f} - 2} < m_2(F).
$$

\hfill \Box
In order to prove Lemma 10 we define a process that generates $F$-closed structures iteratively starting from a single copy of $F$. Our proof simplifies similar approaches from [3] and [5].

Let $G'$ be an $F$-component of the $F$-core of $G_{n,p}$. Then $G'$ can be generated by starting with an arbitrary copy of $F$ in $G'$ and repeatedly attaching copies of $F$ to the graph constructed so far.

Let $F_0$ be a copy of $F$ in $G'$, 
\[ \ell \leftarrow 0; \hat{G} \leftarrow F_0; \]
\[ \text{while } \hat{G} \neq G' \text{ do} \]
\[ \ell \leftarrow \ell + 1; \]
\[ \text{if } \hat{G} \text{ contains an open edge then} \]
\[ \quad \text{let } \ell' < \ell \text{ be the smallest index such that} \]
\[ \quad F_{\ell'} \text{ contains an open edge}; \]
\[ \quad \text{let } e \text{ be any open edge in } F_{\ell'}; \]
\[ \quad \text{let } F_\ell \text{ be a copy of } F \text{ in } G' \text{ that contains } e \text{ but is not contained in } \hat{G}; \]
\[ \quad \text{else} \]
\[ \quad \text{let } F_\ell \text{ be a copy of } F \text{ in } G' \text{ that is not contained in } \hat{G} \text{ and intersects } \hat{G} \text{ in at least one edge;} \]
\[ \hat{G} \leftarrow \hat{G} \cup F_\ell; \]

In order to finally apply a first moment argument we first collect some properties of this process. Consider a copy $F_\ell$ for $\ell \geq 1$. We distinguish two cases: a) $F_\ell$ intersects $\hat{G} := \bigcup_{i<\ell} F_i$ in \textit{exactly} two vertices (that, by definition of the algorithm, have to form an edge), i.e. $F_\ell$ intersects $\hat{G}$ in exactly one edge (we call this a regular copy) and b) $F_\ell$ intersects $\hat{G}$ in some subgraph $J$ with $\nu_J \geq 3$ (we call this a degenerate copy).

In the following we denote the union of the copies $F_0,\ldots,F_\ell$ as the situation at time $\ell$. For $0 \leq i \leq \ell$ we say that the copy $F_i$ is \textit{fully-open} at time $\ell$ if $F_i$ is a regular copy (or $i = 0$) and no vertex of $V(F_i) \setminus \bigcup_{i' < i} V(F_{i'})$, is touched by any of the copies $F_0,\ldots,F_\ell$. Note that $F_0$ is fully-open only at time 0. Also note that, by Lemma 10 every fully-open copy at time $\ell \geq 1$ contains exactly $e_F - 1$ open edges.

For the analysis of the algorithm it is important to keep track of fully-open copies. For doing so we introduce the following definition. For $\ell \geq 1$ let \[ \kappa(\ell) = |\{0 \leq i < \ell \mid F_i \text{ fully-open at time } \ell - 1 \text{ but not at time } \ell\}|. \]

Clearly, a regular copy can 'destroy' at most one fully-open copy (as it intersects $\hat{G}$ in exactly one edge). Thus $\kappa(\ell) \leq 1$ if $F_\ell$ is a regular copy. A degenerate copy on the other hand intersects one $F_i$ in an edge and may destroy up to $e_F - 1$ additional fully-open copies. Thus, $\kappa(\ell) \leq e_F - 1$ if $F_\ell$ is a degenerate copy.

\textbf{Claim 10.} For any sequence $F_i,\ldots,F_{i+e_F-2}$ of consecutive regular copies such that $\kappa(i) = 1$ we have $\kappa(i+1) = \ldots = \kappa(i+e_F-2) = 0$.

\textbf{Proof.} As $F_i$ is a regular copy we know that $F_i$ intersects some copy $F_{i'}$, $i' < i$, in exactly one edge. As $\kappa(i) = 1$ we know that $F_{i'}$ was fully-open at time $i - 1$. Thus at time $i - 1$ the copy $F_{i'}$ had $e_F - 1$ open edges (resp. $e_F$ if $i' = 0$) and the intersection of $F_i$ with $F_{i'}$ is one of these open edges. At time $i + 1$ the copy $F_{i'}$ thus still has at least $e_F - 2$ open edges and since it was chosen by the process at step $i$, it will be chosen again in every consecutive step as long as it has an open edge. It easily follows from Lemma [3] that every regular copy closes at most one open edge, thus each of the copies $F_{i+1},\ldots,F_{i+e_F-2}$ intersects $F_{i'}$ in exactly one open edge, which implies $\kappa(i+1) = \ldots = \kappa(i+e_F-2) = 0$. \qed
Next we estimate the number of fully-open copies at time $\ell$ as a function of the number of regular and degenerate copies. Let us denote with $\text{reg}(\ell)$ and $\text{deg}(\ell)$ the number of copies $F_i$, $1 \leq i \leq \ell$, which are regular, resp. degenerate. Furthermore, we denote with $f_o(\ell)$ the number of fully-open copies at time $\ell$.

**Claim 11.** For every $\ell \geq 1$, assuming the process doesn’t stop before adding the $\ell$-th copy, we have

$$f_o(\ell) \geq \text{reg}(\ell)(1 - 1/(e_F - 1)) - \text{deg}(\ell) \cdot v_F.$$  

**Proof.** Let us denote by $\varphi(\ell) := \text{reg}(\ell)(1 - 1/(e_F - 1)) - \text{deg}(\ell) \cdot v_F$ the right hand side. We use induction to prove the following slightly stronger statement

$$f_o(\ell) \geq \begin{cases} 
\varphi(\ell) & \text{if } F_{\ell} \text{ is a regular copy} \\
\varphi(\ell) + 1 & \text{if } F_{\ell} \text{ is a degenerate copy},
\end{cases}$$

for all $\ell \geq 1$. One easily checks that this claim holds for $\ell = 1$: if $F_1$ is a regular copy then $f_o(1) = 1 > 1 - 1/(e_F - 1)$, otherwise $f_o(1) = 0 > -v_F + 1$. Consider now some $\ell \geq 2$. If $F_{\ell}$ is a degenerate copy then $\kappa(\ell) \leq v_F - 1$ and so $f_o(\ell) = f_o(\ell - 1) - \kappa(\ell) \geq f_o(\ell - 1) - v_F + 1$. The claim thus easily follows from $\text{reg}(\ell) = \text{reg}(\ell - 1)$ and $\text{deg}(\ell) = \text{deg}(\ell - 1) + 1$. Otherwise, assume that $F_{\ell}$ is a regular copy and let

$$\ell' := \max\{1 \leq \ell' < \ell \mid \kappa(\ell') > 0 \text{ or } F_{\ell'} \text{ is a degenerate copy}\}.$$  

Note that $\ell'$ is well defined, as $\kappa(1) = 1$. Note also that the fact that $F_{\ell'}$ is regular together with the definition of $\ell'$ implies that $\varphi(\ell') = \varphi(\ell' + (\ell - \ell')(1 - 1/(e_F - 1)))$. In addition, we deduce from $\kappa(i) = 0$ for $\ell' < i < \ell$ that all steps $\ell' < i < \ell$ add a fully-open copy. We thus have

$$f_o(\ell) = f_o(\ell') + (1 - \kappa(\ell)) + (\ell - \ell' - 1) = f_o(\ell') + \ell - \ell' - \kappa(\ell).$$

If $F_{\ell'}$ is a degenerate copy then the induction assumption implies $f_o(\ell') \geq \varphi(\ell') + 1$. As $F_{\ell'}$ is a regular copy and thus $\kappa(\ell) \leq 1$, this implies $f_o(\ell) \geq \varphi(\ell') + \ell - \ell' \geq \varphi(\ell)$, as claimed. Finally, assume that $F_{\ell'}$ is a regular copy. If $\kappa(\ell) = 0$, then the claim follows trivially from the above inequalities. Otherwise we have $\kappa(\ell) = 1$ and Claim 10 thus implies that $\ell \geq \ell' + (e_F - 1)$. Therefore $f_o(\ell) = f_o(\ell') + \ell - \ell' - 1 \geq f_o(\ell') + (\ell - \ell')(1 - 1/(e_F - 1)) \geq \varphi(\ell)$, similarly as before. \(\square\)

If $f_o(\ell) > 0$ for some $\ell \geq 1$, then $F_{\ell}$ cannot be the last copy in the process because there exist edges which are still open. Furthermore, from Claim 11 we have that after adding $L$ copies, out of which at most $\xi$ were degenerate, there are still at least

$$(L - \xi)(1 - 1/(e_F - 1)) - \xi \cdot v_F$$

fully-open copies at time $L$.

In a first moment calculation we have to multiply the number of choices for $F_{\ell}$ with the probability that the chosen copy of $F$ is in $G_{n,p}$. For a regular copy where $F_{\ell}$ is attached to an open edge we get that this term is bounded by

$$2e_F^2 \cdot n^{v_F - 2} \cdot p^{v_F - 1} \leq 2e_F^2 \cdot c < \frac{1}{2},$$

for $0 < c < 1/(4e_F^2)$. Here the term $2e_F^2$ bounds the number of choices of the open edge in $F_{\ell'}$ (at most $e_F$ choices) times the number of choices for the edge in $F_{\ell}$ that is merged with this open edges ($e_F$ choices) times 2 for the orientation. For a regular copy $F_{\ell'}$ that is attached to a closed edge we also have to replace the first factor $e_F$ by, say, $\ell \cdot e_F$, as the edge $e$ to which the new copy $F_{\ell}$ is attached can be any of the previously added edges. From the above we know that after step $L$ regular copies are always attached to an open edge, as long as the number of degenerate copies is at most $\xi$. That is, in a first moment argument we may bound
the factor attributed to a regular step at time $\leq L$ by $L/2 \leq L$ and after time $L$ by $2^{-1}$.

To bound the term for degenerate copies, observe first that for every subgraph $J \subseteq F$ with $v_J \geq 3$, we have

$$\frac{e_F - 1}{v_F - 2} = m_2(F) \geq \frac{e_J - 1}{v_J - 2} \quad \text{and thus} \quad \frac{e_F - e_J}{v_F - v_J} = \frac{(e_F - 1) - (e_J - 1)}{(v_F - 2) - (v_J - 2)} \geq m_2(F).$$

We may thus choose an $\alpha > 0$ so that

$$(v_F - v_J) - \frac{e_F - e_J}{m_2(F)} < -\alpha \quad \text{for all } J \subseteq F \text{ with } v_J \geq 3.$$

We can now bound the case that the copy $F_\ell$ is a degenerate copy by

$$\sum_{J \subseteq F, v_J \geq 3} (\ell \cdot v_F)^{v_J} \cdot n^{v_F - v_J} \cdot p^{e_F - e_J} \leq (\ell \cdot v_F \cdot 2^{e_F})^{v_F} \cdot n^{-\alpha}, \quad (3)$$

with room to spare.

We now do a union bound. For that we choose $\xi$ such that $\xi \cdot \alpha > v_F + 1$ and then choose $L$ such that the term in $(3)$ is positive. (Observe that $L$ is a constant that only depends on the graph $F$.) Finally, choose $\ell_0 = (v_F + 1) \log_2 n + \xi + L$.

Consider first all sequences of length $\ell \leq \ell_0$ with the property that $F_\ell$ is the $\xi$th degenerate copy. Then the expected number of subgraphs in $G_{n,p}$ that can be built by such a sequence is at most

$$\sum_{\ell \leq \ell_0} \frac{1}{(\ell - 1)!} n^{v_F \cdot \left[(\ell_0 v_F 2^{e_F})^{v_F} \cdot n^{-\alpha}\right]^{\xi} \cdot L \cdot 2^{-(\ell - L - \xi)} \leq n^{v_F} \cdot o(n) \cdot n^{-\alpha} \cdot \ell = o(1),$$

by choice of $\xi$. Here the binomial coefficient corresponds to the choices (in time) when the first $\xi - 1$ degenerate steps occurred. The term $n^{v_F}$ bounds the choice of the copy $F_0$. The next factor bounds the choices of degenerate steps, the last two factors bound those of the regular steps: as explained above, regular copies contribute a term of at most $L$ if they occur before step $L$ and a factor $1/2$ if they occur after step $L$.

So we know that within the first $\ell_0$ copies we have less than $\xi$ degenerate ones. Observe that by the choice of $L$ this implies that the sequence that generates $G'$ either has length less than $L$ (which is fine) or length at least $\ell_0$. The latter follows from the fact that $[1]$ is increasing in $L$, and thus $f_0(\ell') > 0$ for every $L < \ell' \leq \ell_0$. It thus suffices to consider all sequences of length $\ell_0$. The expected number of subgraphs in $G_{n,p}$ that can be built by such a sequence is at most

$$\sum_{k \leq k_0} \frac{\ell_0}{k} n^{v_F} \cdot [(\ell_0 v_F 2^{e_F})^{v_F} \cdot n^{-\alpha}]^{k} \cdot L \cdot 2^{-(\ell_0 - L - k)} \leq n^{v_F} \cdot o(n) \cdot n^{-v_F + 1} \cdot L = o(1),$$

by choice of $\ell_0$. This concludes the proof of Lemma 6 and thus also the proof of the 0-statement.

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References

Our proof essentially follows the approach in [9].


For convenience of the reader we provide in this appendix the proof of the deterministic statement (Theorem 7) that was used in the proof of the 0-statement. Our proof essentially follows the approach in [9].

Using standard notation, we denote with \( m(G) = \max_{J \subseteq G} \frac{\delta(J)}{\varepsilon} \) the maximum density of a graph \( G \), and with

\[
\ar(G) = \max_{J \subseteq G, v_j \geq 2} \frac{\varepsilon_j}{v_j - 1}
\]

the arboricity of the graph \( G \). One easily checks that all graphs \( G \) satisfy

\[
m(G) \leq \ar(G) \leq m(G) + \frac{1}{2}.
\]

Let \( \delta(G) \) denote the minimum degree in \( G \), i.e., \( \delta(G) := \min_{v \in V(G)} \deg(v) \). Furthermore, let \( \delta_{\max}(G) := \max_{G' \subseteq G} \delta(G') \) be the maximum minimum degree in all subgraphs of \( G \).

The following lemma gives various conditions under which \( G \not\rightarrow (F)_{\varepsilon_2} \).

**Lemma 12.** Let \( G \) and \( F \) be two graphs such that at least one of the following properties is satisfied:

(i) \( \ar(G) \leq 2 \cdot |\ar(F) - \varepsilon| \) for some \( \varepsilon > 0 \),

(ii) \( m(G) \leq 2 \cdot m(F) - \varepsilon \) for some \( \varepsilon > 0 \),

(iii) \( \delta_{\max}(G) \leq 2(\delta(F) - 1) \),

(iv) \( m(G) < \delta_{\max}(F) \) and \( \chi(F) \geq 3 \).

Then \( G \not\rightarrow (F)_{\varepsilon_2} \).

**Proof.** Assume first that (i) holds. Nash-Williams’ Arboricity Theorem (cf. [2] for a short and self-contained proof) states that for every graph \( G = (V, E) \) there exists a partition of the edges into \( |\ar(G)| \) parts, \( E = E_1 \cup \ldots \cup E_{|\ar(G)|} \), such that all \( E_i \) are forests. If we thus colour the edges in the first \( |\ar(F) - \varepsilon| \) of these sets red and the remaining edges blue, then both the blue as well as the red subgraph have arboricity at most \( |\ar(F) - \varepsilon| < \ar(F) \) and thus cannot contain a copy of \( F \).

In case of (ii) we proceed similar. We replace Nash-Williams’ theorem by the following statement: for every graph \( G = (V, E) \) there exists a partition of the edges into \( |m(G)| \) parts, \( E = E_1 \cup \ldots \cup E_{|m(G)|} \), such that all components in \( E_i \) contain at most one cycle. (This follows easily from Hall’s theorem applied to a bipartite graph with \( |m(G)| \) copies of every vertex in one set and one copy of every edge in the other set.) Similarly as before we now colour the edges in the first \( m(F) - \varepsilon \) of these sets red and the remaining edges blue. Then subgraphs have maximum density at most \( m(F) - \varepsilon \) and thus cannot contain a copy of \( F \).

Now assume that (iii) holds. Construct a sequence \( v_1, v_2, \ldots, v_{|\ar(G)|} \) of the vertices in \( G \) as follows: let \( v_1 \) be a vertex of minimum degree in \( G \) \( - \{v_1, \ldots, v_{i-1}\} \). Then

\[
\delta_2(G) := \min_{v \in V(G)} \delta_2(G) \leq \delta_2(F) - 1
\]

and

\[
\delta_3(G) := \min_{v \in V(G)} \delta_3(G) \leq \delta_3(F) - 1
\]

are forests. If we thus colour the edges in the first

\[
|\ar(G)| \geq |\ar(F)| - \varepsilon
\]

of these sets red and the remaining edges blue, then both the blue as well as the red subgraph have

\[
\ar(G) \leq \ar(F) - \varepsilon
\]

and thus cannot contain a copy of \( F \).
every vertex $v_i$ has degree at most $\delta_{max}(G)$ into $G([v_{i+1}, \ldots, v_{v_g}])$, the graph
induced by the vertices 'to the right'. Color the vertices of $G$ 'backwards', i.e.,
starting with $v_{v_g}$ (which is coloured arbitrarily). As every vertex $v_i$ has degree at
most $\delta_{max}(G) \leq 2(\delta(F) - 1)$ into the part that is already coloured, we can colour
$\delta(F) - 1$ of these edges blue and the remaining ones red. Clearly, the coloured part
can then not contain a monochromatic copy of $F$ that contains $v_i$. By repeating
this procedure for every vertex $v_i$ we obtain a colouring without a monochromatic
$F$.

Finally, assume that (iv) holds. Here we proceed in two steps. We first show
that we can find a vertex-colouring of $G$ without a monochromatic copy of $F$. Then
we show that this implies that there also exists an edge-colouring of $G$ without a
monochromatic copy of $F$. Indeed, the latter is easy. Assume there exists a 2-
colouring of the vertices, that is, a partition $V = X \cup Y$ such that neither $G[X]$ nor
$G[Y]$ contains a copy of $F$. Then we colour all edges in $G[X]$ and $G[Y]$ with red and
all edges in $E(X, Y)$ with blue without inducing a monochromatic copy of $F$ (as $F$ is
connected and is not bipartite). So we need to find the vertex-colouring. The
following argument is from [7]. Let $F' \subset F$ such that $\delta(F') = \delta_{max}(F)$. We need
to show that for every graph $G$ with $m(G) < \delta(F')$ we find a vertex colouring of $G$
without a monochromatic $F'$. Assume this is not true. Then there exists a minimal
counterexample $G_0$. As $G_0$ is minimal we know that for every vertex $v \in V(G_0)$
the graph $G_0 - v$ does have a vertex colouring without a monochromatic $F'$. Clearly,
if $deg(v) < 2\delta(F')$ then such a colouring can be extended to $v$. So we know that in
$G_0$ every vertex has degree at least $2\delta(F')$, that is $m(G_0) \geq \sum_v deg(v)/(2\delta_{2n}) \geq
\delta(F') = \delta_{max}(F)$. A contradiction.

Let $G'$ be a subgraph of $G$ with minimum degree $\delta_{min}(G)$. Then $|E(G')| \geq
\frac{1}{2} |V(G')| \cdot \delta_{max}(G)$ and we thus see that

$$2m(G) \geq \delta_{max}(G). \quad (5)$$

We use this observation to show that graphs with density at most 2 can be coloured
without a monochromatic triangle.

**Lemma 13.** Let $G$ be a graph such that $m(G) \leq 2$. Then $G \not\rightarrow (K_3)^2_5$.

**Proof.** We proceed similarly as in the previous proof. We construct a sequence
$v_1, v_2, \ldots$, by choosing $v_i$ as a vertex of minimum degree in $G - \{v_1, \ldots, v_{i-1}\}$, with
the additional condition that the neighborhood of $v_i$ in $G - \{v_1, \ldots, v_{i-1}\}$
is not a $K_4$. If we do not find a vertex that satisfies this property then we stop.
As $\delta_{max}(G) \leq 2m(G) \leq 4$ we will always find a vertex with degree at most four.
Also note that if the minimum degree is four, then the graph is 4-regular. That is,
the above process can only stop if every vertex has degree 4 and has the property
that its neighborhood induces a $K_4$. In other words, if we cannot find a vertex
$v_i$, then $G' := G - \{v_1, \ldots, v_{i-1}\}$ is a union of vertex-disjoint $K_5$'s. Since there
exists a 2-colouring of a $K_5$ without a monochromatic triangle, we can thus 2-
colour $G'$ without a monochromatic triangle. Now we proceed again as in the
previous proof and colour the remaining vertices in reverse order. By construction
vertex $v_i$ has degree at most 4 into $G([v_i, \ldots, v_{v_g}])$ and the neighbourhood of $v_i$
in $G([v_i, \ldots, v_{v_g}])$ is not a $K_4$. A simple case checking shows that however the
neighborhood of $v_i$ is coloured without a monochromatic triangle there is always
an extension of the colouring to the edges incident to $v_i$ so that no monochromatic
triangle is generated. □

**Proof of Theorem [7]** Observe that for $v_F = 3$ the only graph with $m_2(F) > 1$ is the
triangle, for which the claim of the theorem holds according to Lemma 13. In the
remainder of the proof we thus assume $v_F \geq 4$. Observe that we may also assume without loss of generality that $F$ is strictly 2-balanced.

The assumption that $F$ is strictly 2-balanced implies that $m_2(F) = \frac{e_F - \delta(F) - 1}{v_F - 3}$ from which we deduce $m_2(F) < \delta(F) \leq \delta_{\max}(F)$. Thus if $\chi(F) \geq 3$, then $F$ satisfies property $\text{(iv)}$ of Lemma $12$ which implies $G \not\rightarrow (F)_2^\ast$. Therefore, in the following, we assume that $F$ is bipartite. Then $e_F \leq \frac{1}{4}v_F^2$ implies that $m_2(F) \leq m(F) + \frac{1}{2}$ with equality if and only if $e_F = \frac{1}{4}v_F^2$.

If $m_2(F) = k + x$ for some $k \in \mathbb{N}$ and $\frac{1}{2} \leq x < 1$ we thus have $m(F) > k$ whenever $x > \frac{1}{2}$ or $e_F < \frac{1}{4}v_F^2$. In this case we have $m(G) \leq k + 1 \leq 2k = 2\lfloor m(F) - \varepsilon \rfloor$ and $F$ satisfies property $\text{(ii)}$ of Lemma $12$ which concludes the proof of the theorem in this case. So we may assume that $x = \frac{1}{2}$ and $e_F = \frac{1}{4}v_F^2$. Then, $v_F = 2\ell$ for some $\ell \in \mathbb{N}$, and thus $m_2(F) = (\ell^2 - 1)/(2\ell - 2) = \frac{1}{2}(\ell + 1)$. That is, $k = \frac{1}{2}\ell$ and so $ar(F) = e_F/(v_F - 1) > \frac{1}{2}v_F = k$. By $4$ we also have $ar(G) \leq m(G) + \frac{1}{2} \leq m_2(F) + \frac{1}{2} = k + 1$. Thus $ar(G) \leq k + 1 \leq 2k \leq 2\lfloor ar(F) - \varepsilon \rfloor$ and $F$ satisfies property $\text{(i)}$ of Lemma $12$ which concludes the proof of the theorem for this case.

Finally, if $m_2(F) = k + x$ for some $k \in \mathbb{N}$ and $0 \leq x < 0.5$ then $5$ implies that $\delta_{\max}(G) \leq 2m(G) \leq 2m_2(F)$ and thus $\delta_{\max}(G) \leq 2k$, as $\delta_{\max}(G)$ is integral. On the other hand, we have already shown that the assumption that $F$ is strictly 2-balanced implies that $m_2(F) < \delta(F)$. The fact that $\delta(F)$ is integral thus implies $\delta(F) \geq k + 1$, and $F$ satisfies property $\text{(iii)}$ of Lemma $12$ which concludes the proof of the theorem for this case. □