Random Graphs

Exercise 11

1. Let \( n \) be sufficiently large and let \( 0 < d < n - 1 \) be an integer. Prove that a.s. \( G(n, d/n) \) is not \( d \)-regular.

2. Let \( G = G_{k-out} = (V, E) \), and let \( u \in V \) be an arbitrary vertex.
   
   (a) What is the expected in-degree of \( u \) in \( G \)?
   
   (b) What is the expected number of pairs of anti-parallel edges (that is, pairs \( \{u, v\} \in \binom{[n]}{2} \) such that \( (u, v) \in E \) and \( (v, u) \in E \))? 

3. Let \( B_{k-out} \) denote the random bipartite directed graph, with both parts of size \( n \), in which every vertex chooses \( k \) out-neighbors uniformly at random, without replacement, and independently of every other vertex. Prove that a.s. \( B_{k-out} \) admits a perfect matching, for every \( k \geq 10 \).

   **Hint:** Denote the parts by \( A \) and \( B \). By Hall’s Theorem, if the graph does not admit a perfect matching, then there exists a subset \( X \subseteq A \) with \( |\Gamma(X)| < |X| \). Consider the following two cases separately:

   (a) \( 1 \leq |X| \leq n/2 \);

   (b) \( n/2 < |X| \leq n \).

   Note that the lower bound \( k \geq 10 \) is not tight.

**Remark:** In all questions concerning \( G_{k-out} \) (or \( B_{k-out} \)) you can assume that \( k \) is a constant.