Random Graphs

Exercise 5

1. For every positive integer $k$, let $X^k$ be a random variable that counts the number of $k$-cliques in $G(n, 1/3)$, and let $f_n(k) = \mathbb{E}(X^k)$.

   (a) Prove that there is a unique positive integer $q$ such that $f_n(q) \geq 1 > f_n(q+1)$. What is (approximately) the value of this $q$?

   (b) Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing series of integers, such that for every $n \in \mathbb{N}$ there exists an integer $q$ for which $f_{a_n}(q) = 3$. Prove that

$$\lim_{n \to \infty} Pr(q - 1 \leq \omega(G(a_n, 1/3)) \leq q) = 1.$$ 

   **Hint:** For every integer $2 \leq i \leq t$, let $g(i) = \frac{\binom{i}{2}(n-i)}{\binom{n}{i}} \left(3\binom{i}{2} - 1\right)$. You can use without a proof the fact that $g(i) \leq \max\{g(2), g(t)\}$ for every integer $2 \leq i \leq t$.

2. Let $G = (V, E)$ be a graph with chromatic number $\chi$ and independence number $\alpha$. Prove that $|V| \leq \chi \alpha$. 
