Graphs and Algorithms

Edge Covers and Matchings

An edge cover of a graph \( G = (V, E) \) is a set \( C \subseteq E \) of edges such that every vertex of \( G \) is incident to some edge of \( C \). Let \( C^* \) be a minimum edge cover of \( G \) and \( M^* \) be a maximum matching of \( G \). Show that \( |C^*| + |M^*| = |V| \).

Sperner’s Lemma

Prove Sperner’s lemma: Given an \( n \)-element set \( X \), there are at most \( \binom{n}{\lfloor n/2 \rfloor} \) subsets of \( X \) such that no two contain each other.

Hint: Consider the \( n \)-dimensional hypercube. We call a directed path \( (v_1, v_2, \ldots, v_k) \) in the hypercube monotone if and only if for any two vertices \( v_i, v_j \) with \( 1 \leq i < j \leq k \), the vertex label of \( v_j \) can be obtained from \( v_i \) by flipping 0-coordinates to 1-coordinates. Construct \( \binom{n}{\lfloor n/2 \rfloor} \) monotone paths covering the \( n \)-dimensional hypercube.

Cantor-Bernstein-Schröder Theorem

Prove the Cantor-Bernstein-Schröder theorem: If there exist injective functions \( f : A \to B \) and \( g : B \to A \) between two infinite sets \( A \) and \( B \), then there exists a bijective function \( h : A \to B \).

Hint: Consider alternating chains of elements obtained by repeatedly applying \( f, g \) and their inverse functions.

Discussion of the exercises on 03.04.2008.