Coloring Outerplanar Graphs

A graph is called outerplanar if it has a planar embedding in the plane such that all vertices lie on the outer (unbounded) face. Without using the Four Color Theorem, prove that every outerplanar graph is 3-colorable. Apply this to prove the Art Gallery Theorem: If an art gallery is laid out as a simple polygon with \( n \) sides (see Figure 1 for an example), then it is possible to place \( \lfloor n/3 \rfloor \) guards on the boundary or the interior of the polygon such that every point of the interior is visible to some guard. Construct a polygon that requires \( \lfloor n/3 \rfloor \) guards.

![Simple Polygon](Figure 1: Simple Polygon)

Characterization of Outerplanar Graphs

Show that a graph \( G \) is outerplanar iff it contains no subdivision of \( K_4 \) or \( K_{2,3} \) as a subgraph. For your proof, you can use Kuratowski’s Theorem which states that a graph is planar iff it does not contain a subdivision of \( K_5 \) or \( K_{3,3} \) as a subgraph.

Tutte’s Theorem

In the lecture, the conflict graph of a cycle of a graph has been defined. Prove Tutte’s Theorem, i.e., prove that a graph is planar iff the conflict graphs of all its cycles are bipartite.

Remark: Tutte’s Theorem is an important step in proving the correctness of one of the planarity tests presented in the lecture. Its proof is not straightforward however, and this problem is included for interested students.