Graphs and Algorithms

Greedy Coloring of Trees

We construct an infinite family of rooted trees $T_k$ ($k \geq 0$) as follows: Let $T_0$ be an isolated vertex and $T_k$ the tree that is obtained by connecting a new root vertex $v$ to the root vertices of copies of all trees $T_0, T_1, \ldots, T_{k-1}$. Clearly we have $n(T_k) = 2^k$. We want to show that there is an ordering $\pi$ of the vertices of $T_k$ such that $\text{Greedy-Coloring}(T_k, \pi) = k + 1 = \Omega(\log n)$. Such an ordering $\pi$ can be defined recursively.

Maintain Connectivity

Let $v$ be a vertex with maximum degree. Then $H := G[\{v\} \cup \Gamma(v)]$ is not a complete graph, otherwise $G = H$ would be a complete graph or $H$ would be a component of $G$ disconnected from the rest of $G$. Hence we can find vertices $\hat{x}$ and $\hat{y}$ in $\Gamma(v)$ with $\text{dist}(\hat{x}, \hat{y}) = 2$. If $G - \hat{x} - \hat{y}$ is connected then we are done. Otherwise, $\{\hat{x}, \hat{y}\}$ is a minimal separating set. As $G$ is 2-connected and not a cycle, we have $\Delta(G) \geq 3$ and hence $\text{deg}(v) \geq 3$. This means the component $C$ of $G - \hat{x} - \hat{y}$ that contains $v$ contains more vertices apart from $v$. As $v$ is not an articulation node in $G$, there is a vertex $x$ from $C - v$ that is a neighbor of $\hat{x}$ or $\hat{y}$. As $\{\hat{x}, \hat{y}\}$ is a minimal separating set, every component of $G - \hat{x} - \hat{y}$ contains neighbors from both $\hat{x}$ and $\hat{y}$. In particular there is a vertex $y$ with $\text{dist}(x, y) = 2$ from a component in $G - \hat{x} - \hat{y}$ different from $C$. We claim that $G - x - y$ is connected. As $\hat{x}$ and $\hat{y}$ are connected in $G - x - y$ via the vertex $v$, it suffices to show that every vertex in $G - x - y$ is connected to $\hat{x}$ or $\hat{y}$. As $G - x$ is connected, every vertex in $C - x$ is connected in $G - x$ to $\hat{x}$ or $\hat{y}$ via a path that does not contain $y$. Similarly, as $G - y$ is connected, every vertex in $(V(G) \setminus C) - y$ is connected in $G - y$ to $\hat{x}$ or $\hat{y}$ via a path that does not contain $x$. 