Monotone Subsequences

The sequence \( (i - 2(n - 1)\left\lfloor \frac{l + 1}{n-1} \right\rfloor)_{i=1, \ldots, (n-1)^2} \) has length \( (n - 1)^2 \) and obviously it does not contain a monotone subsequence of length \( n \). If we can prove that every sequence of \( (n - 1)^2 + 1 \) pairwise distinct numbers contains a monotone subsequence of length \( n \), then we have shown that \( N = N(n) = (n - 1)^2 + 1 \).

We proceed by induction on \( n \). The induction basis \( n = 1 \) is clear: Every sequence of length 1 contains a monotone subsequence of length 1.

Assuming that the statement holds for \( n \geq 1 \) we now show that it also holds for \( n + 1 \). Given a sequence \( (a_i) \) of length \( N(n+1) = n^2 + 1 \), we apply the induction hypothesis repeatedly to obtain \( N(n+1) - N(n) = 2n \) different first elements of decreasing subsequences of length \( n \) and last elements of increasing subsequences of length \( n \). Let \( a_{i_1}, a_{i_2}, \ldots, a_{i_k} \) \((i_1 < i_2 < \cdots < i_k)\) be the first elements of decreasing subsequences and \( a_{j_1}, a_{j_2}, \ldots, a_{j_l} \) \((j_1 < j_2 < \cdots < j_l)\) the last elements of increasing subsequences \((k + l = 2n)\). If one of the subsequences starting with \( a_{i_1}, a_{i_2}, \ldots, a_{i_k} \) can be extended to the left by a larger element, then we have found a decreasing subsequence of length \( n + 1 \) and we are done. If one of the subsequences ending with \( a_{j_1}, a_{j_2}, \ldots, a_{j_l} \) can be extended to the right by a larger element, then we have found an increasing subsequence of length \( n + 1 \) and we are done. Otherwise we have \( a_{i_1} < a_{i_2} < \cdots < a_{i_k} \) and \( a_{j_1} > a_{j_2} > \cdots > a_{j_l} \) and \( i_k < j_1 \). If \( k > n \), then \( a_{i_1}, a_{i_2}, \ldots, a_{i_k} \) form an increasing subsequence of length at least \( n + 1 \) and we are done. If \( l > n \), then \( a_{j_1}, a_{j_2}, \ldots, a_{j_l} \) form a decreasing subsequence of length at least \( n + 1 \) and we are done. Otherwise we conclude from \( k \leq n, l \leq n \) and \( k + l = 2n \) that \( k = l = n \). If \( a_{i_k} < a_{j_1} \), then \( a_{i_1}, a_{i_2}, \ldots, a_{i_k}, a_{j_1} \) form an increasing subsequence of length \( n + 1 \), otherwise \( a_{i_k} > a_{j_1} \) and \( a_{i_1}, a_{i_2}, \ldots, a_{i_k}, a_{j_1} \) form a decreasing subsequence of length \( n + 1 \).

**Remark:** Let \( N(n, n') \) denote the smallest number such that every sequence of \( N(n, n') \) pairwise distinct numbers contains an increasing subsequence of length \( n \) or a decreasing subsequence of length \( n' \). From our previous considerations we know that \( N(n, n) = (n - 1)^2 + 1 \) and it is not hard to generalize the proof to \( N(n, n') = (n - 1)(n' - 1) + 1 \).

Convex Point Sets

First observe that for any set of points in the plane we can fix a coordinate system such that no two points share the same abscissa and no two points share the same ordinate. We call a sequence of points \( A_1, A_2, \ldots, A_k \) with increasing abscissa values **convex** if the gradients of the segments \( A_1A_2, A_2A_3, \ldots, A_{k-1}A_k \) are monotonously decreasing and **concave** if the gradients are monotonously increasing. Let \( N(i, k) \) denote the smallest number such that among any set of \( N(i, k) \) points we can find a convex sequence of \( i \) points or a concave sequence of \( k \) points. We claim that

\[
N(i, k) \leq N(i - 1, k) + N(i, k - 1) - 1.
\]

The reasoning is very similar to the first problem. Consider a set \( S \) of \( N(i - 1, k) + N(i, k - 1) - 1 \) points in the plane and a subset \( S' \subseteq S \) of \( N(i - 1, k) \) points. If \( S' \) contains a concave sequence of \( k \) points then we are done. Otherwise it contains a convex sequence of \( i - 1 \) points. Exchange
the last point from this sequence for another element in $S \setminus S'$ and repeat the same argument. In this way we either find a concave sequence of $k$ points or a set $S''$ of $N(i, k - 1)$ endpoints of convex sequences of $i - 1$ points each. If $S''$ contains a convex sequence of $i$ points then we are done. Otherwise it contains a concave sequence $B_1, B_2, \ldots, B_{k - 1}$ of $k - 1$ points. We know that $B_1$ is the endpoint of a convex sequence $A_1, A_2, \ldots, A_{i - 1}$ of $i - 1$ points ($A_{i - 1} = B_1$). If the gradient of $A_{i - 2}A_{i - 1}$ is greater or equal than the gradient of $B_1B_2$ then we have found a convex sequence $A_1, A_2, \ldots, A_{i - 1}, B_2$ of $i$ points. Otherwise we have found a concave sequence $A_{i - 2}, B_1, B_2, \ldots, B_{k - 1}$ of $k$ points.

Using that $N(3, k) = N(k, 3) = k$ we obtain $N(k, k) \leq \binom{2k - 4}{k - 2} + 1$. Observe that if we can find a convex or concave subsequence of $k$ points in a set of $N(k, k)$ points in the plane, then by connecting the first and the last point of this sequence we have found a convex polygon on $k$ vertices.