Exercise 1 (Edge Covers and Matchings)

An edge cover of a graph $G = (V, E)$ is a set $C \subseteq E$ of edges such that every vertex of $G$ is incident to at least one edge of $C$. Let $C^*$ be a minimum edge cover of $G$ (i.e., an edge cover of minimum cardinality) and $M^*$ be a maximum matching of $G$. Show that $|C^*| + |M^*| = |V|$.

Exercise 2 (Independent Sets and Edge Covers)

An independent set in a graph $G = (V, E)$ is a set $I \subseteq V$ of vertices such that no two vertices in $I$ are adjacent. Let $I^*$ be a maximum independent set of $G$ and $C^*$ a minimum edge cover.

(a) Prove that if $G$ is connected we have $|C^*| \geq |I^*|$.

(b) Prove that if $G$ is connected and bipartite we have $|C^*| = |I^*|$. 

Remark: In particular this shows that $|I^*| + |M^*| = |V|$ if $G$ is connected and bipartite.

Exercise 3 (Hall’s Theorem on Infinite Graphs)

Find a counterexample to Hall’s Theorem on infinite graphs. That is, find a bipartite graph $G = (A \cup B, E)$ where $A$ and $B$ have more than finitely many vertices, such that $|S| \leq |N(S)|$ for all $S \subseteq A$, but there is no matching of cardinality $A$ in $G$. 