This is the first special (i.e., graded) exercise set. Please send your solutions to lthomas@inf.ethz.ch.

**Regulations:**

- There will be a total of four special exercise sets during this semester.
- You are expected to solve them carefully and then write a nice and complete exposition of your solutions using LaTeX.
- You are welcome to discuss the exercises with your colleagues, but we expect each of you to hand in your own, individual writeup.
- Your solutions will be graded. The three highest out of your four achieved grades will each account for 10% of your final grade for the course (so 30% of the grade in total).

**Exercise 1**

(8 points)

Show that for every integer sequence \(d_1, d_2, \ldots, d_n\) with \(d_i \geq 1\) for all \(1 \leq i \leq n\) and \(\sum_{i=1}^{n} d_i = 2(n - k)\) for some positive integer \(k\), there exists a forest on the vertex set \(\{1, \ldots, n\}\) with \(k\) components, satisfying \(\text{deg}(i) = d_i\) for all \(1 \leq i \leq n\).

**Exercise 2**

(4 + 4 points)

(a) For any \(k \geq 2\), show that in a \(k\)-connected graph every \(k\) vertices lie on a common cycle.

(b) Given a \(d\)-regular, \(d\)-edge-connected graph \(G\) for some integer \(d\), prove that for every \(k \leq n\) removing \(k\) vertices from \(G\) results in a graph with at most \(k\) components.

**Exercise 3**

(8 points)

Two people play a game on a graph \(G\), alternately choosing previously unclaimed vertices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player). Thus together the two players follow a path. The last player able to move wins. Prove that Player 2 has a winning strategy if \(G\) has a perfect matching, and otherwise Player 1 has a winning strategy.

**Exercise 4**

(4 + 4 points)

(a) Prove that any \(k\)-regular bipartite graph is the union of \(k\) perfect matchings.

(b) A *latin rectangle* is a grid with \(m\) rows and \(n\) columns (where \(m < n\)) containing elements from \(\{1, \ldots, n\}\) such that in each row and in each column no element occurs more than once (see Figure 1). A *latin square* is a grid of \(n\) rows and \(n\) columns with these properties. Prove that any latin rectangle can be completed to a latin square.
2 5 3 1 4
3 4 2 5 1
1 2 4 3 5
5 3 1 4 2
4 1 5 2 3

Figure 1: A latin rectangle for \( m = 3 \) and \( n = 5 \) with a completion to a latin square

**Exercise 5**

\((8 \text{ points})\)

A \( k \)-orientation of a graph \( G \) is an orientation of all edges in \( G \) such that the in-degree of every vertex (that is, the number of edges oriented towards a vertex) is at most \( k \). \( G \) is called \( k \)-orientable if there exists a \( k \)-orientation of \( G \).

The edge-density of a graph \( G \) is defined by \( d(G) := |E(G)|/|V(G)| \). The maximum edge-density of \( G \) is defined by

\[
m(G) := \max_{H \subseteq G, V(H) \neq \emptyset} d(H).
\]

Prove that a graph \( G \) is \( k \)-orientable if and only if \( \lceil m(G) \rceil \leq k \).

*Hint:* Construct a bipartite graph \( B \) with \( E(G) \) as one part such that a matching of cardinality \( |E(G)| \) in \( B \) corresponds to a \( k \)-orientation of \( G \).

**Submission due on 25.03.2009, 23:59.**