Graphs and Algorithms

Exercise 1

The Kneser graph $KG_{n,k}$ is defined on the vertex set $V(KG_{n,k}) := \binom{[n]}{k}$ where for every $A, B \in \binom{[n]}{k}$ we have that $\{A, B\} \in E(KG_{n,k})$ if and only if $A \cap B = \emptyset$. (Observe that $KG_{5,2}$ is isomorphic to the Petersen graph.)

Prove that $KG_{n,2}$ contains a Hamilton cycle for every $n \geq 8$.

Exercise 2

Prove or disprove the following statements:

(a) The edges of any 4-regular graph can be colored with two colors, say red and blue, such that the blue edges form a 2-regular graph and the red edges form a 2-regular graph.

(b) The edges of any 6-regular graph can be colored with two colors, say red and blue, such that the blue edges form a 3-regular graph and the red edges form a 3-regular graph.

Exercise 3

Let $G$ be a directed graph on the vertex set $V := \{1, \ldots, n\}$. For every $s, t \in V$ (where $s \neq t$) let $X_{s,t}$ denote the number of directed Hamiltonian paths from $s$ to $t$ in $G$, i.e. $s-t$-paths that visit every vertex exactly once. For any subset $S \subseteq V \setminus \{s, t\}$ let $N_{s,t}(S)$ denote the number of walks of length $n-1$ from $s$ to $t$ containing no vertex in $S$.

(a) Show that $X_{s,t} = \sum_{S \subseteq V \setminus \{s, t\}} (-1)^{|S|} N_{s,t}(S)$.

(b) Show that $N_{s,t}(S)$ can be computed in polynomial time via dynamic programming.

(c) Use (a) and (b) to obtain a polynomial space algorithm that decides if $G$ contains a Hamiltonian cycle. What is the running time of your algorithm?
Exercise 4

Remark: The Association of Computing Machinery (ACM) runs an archive of programming challenges on their website http://icpcres.ecs.baylor.edu/onlinejudge/. Solutions to a problem can be verified by an online judge. The following exercise is a reformulation of problem 10630 Millennium Ceremony (Contest Volume CVI). You can either solve the exercise as stated below for a total of 10 points or you can get up to 15 points if you write a program which passes the tests of the online judge. In the latter case your solution should consist of the commented source code as well as the evaluation by the online judge.

The following construction is illustrated in Figure 1. We define the \( P(n, m, k) \) for any even \( m \) and any odd \( k \) as follows. First we define for all \( 1 \leq i \leq n \) a graph \( G_i = (V_i, E_i) \) with \( V_i := \{i\} \times [m] \) and \( E_i := \{(i, j), (i, (j \mod m) + 1)\} : 1 \leq j \leq m \}, \) i.e., the graph \( G_i \) is a cycle on \( m \) vertices labeled \((i, 1), (i, 2), \ldots, (i, m)\). Furthermore, we let \( G_i^{i+1} := (V_i \cup V_{i+1}, E_i^{i+1}) \) be an arbitrary \( k \)-regular bipartite graph (with parts \( V_i \) and \( V_{i+1} \)) such that for every edge \( \{(i, a), (i + 1, b)\} \in E_i^{i+1} \) we have \( a \equiv b \mod 2 \). (Observe that \( G_i^{i+1} \) thus is the union of two \( k \)-regular bipartite graphs, one on the “even” vertices and one on the “odd” ones.)

\[
V(P(n, m, k)) := \{b, t\} \cup \bigcup_{i=1}^{n} V(G_i),
\]
\[
E_b := \{\{b, v\} : v \in V(G_1)\} \quad \text{(where } b \text{ stands for ‘base’)} \cup \bigcup_{i=1}^{n-1} E_i,
\]
\[
E_t := \{\{v, t\} : v \in V(G_n)\} \quad \text{(where } t \text{ stands for ‘top’}) \cup \bigcup_{i=1}^{n-1} E_i^{i+1},
\]
\[
E(P(n, m, k)) := E_b \cup E_t \cup \bigcup_{i=1}^{n} E_i \cup \bigcup_{i=1}^{n-1} E_i^{i+1}.
\]

Prove that it is always possible to find two Euler tours \( ET_r, ET_b \) starting and ending at \( b \) with the following properties:
$ET_r$ and $ET_b$ induce a coloring on the edges of $P(n, m, k)$ as follows: In the beginning every edge is colored black. We “run” $ET_r$ and if it visits a vertex $v \in V(P(n, m, k))$ for the first time then the edge through which it came to the vertex $v$ is colored red, otherwise the edge is left black. The coloring induced by the complete tour $ET_r$ restricts $ET_b$ as follows: Whenever $ET_b$ visits a vertex $v$ for the first time it must approach $v$ with a black edge. (That is, every edge colored red by $ET_r$ cannot be used by $ET_b$ to visit a new vertex.) We now “run” $ET_b$ and analogically, if $ET_b$ visits a vertex $v \in V(P(n, m, k))$ for the first time then the (black) edge through which it came to the vertex $v$ is colored blue.

Now the property asked for is that after “running” $ET_r$ and $ET_b$, in every layer $1 \leq i \leq n$ the edges $E(G_i)$ are alternately colored red and blue.

**Hint:** Think of what you know about $k$-regular bipartite graphs. Construct two (not necessarily closed) tours that visit every vertex and ensure the coloring condition and that use some, but not necessarily all of $P(n, m, k)$’s edges. Then complete them to closed Euler tours.

**Submission due on 29.04.2009, 23:59.**