Exercise 1 (Edge Covers and Matchings)

From a maximum matching $M^*$ we will construct an edge cover of size $|V| - |M^*|$. Since a smallest edge cover is not bigger than this cover, this implies $|C^*| \leq |V| - |M^*|$. Also, from a minimum edge cover $C^*$ we will construct a matching of size $|V| - |C^*|$. Since a largest matching is not smaller than this matching, this implies $|M^*| \geq |V| - |C^*|$. These two inequalities prove that $|C^*| + |M^*| = |V|$.

Consider a maximum matching $M^*$. We construct an edge cover $C$ of $G$ as follows: We start with $C := M^*$ and for each of the $|V| - 2|M^*|$ vertices which are not saturated by $M^*$ we add an incident edge to $C$. Hence we have $|C| = |M^*| + |V| - 2|M^*| = |V| - |M^*|$.

Now consider a minimum edge cover $C^*$. Let $S_1, \ldots, S_s$ denote the components formed by the edges of $C^*$. Clearly, $C^*$ does not contain cycles, and all paths in $C^*$ have length at most 2 since otherwise we can construct an edge cover of smaller size. Hence, $S_i$ is a star for every $1 \leq i \leq s$. We now build a matching $M$ by choosing an arbitrary edge from every star, and obtain

$$|M| = \sum_{i=1}^{s} 1 = \sum_{i=1}^{s} |(V(S_i) - E(S_i))| = \sum_{i=1}^{s} |V(S_i)| - \sum_{i=1}^{s} |E(S_i)| = |V| - |C^*|.$$

Exercise 2 (Independent Sets and Edge Covers)

(a) Let $I^*$ be a maximum independent set and $C^*$ be a minimum edge cover. Clearly, every edge in $C^*$ contains at most one vertex of $I^*$ and since $G$ is connected, every vertex in $I^*$ is contained in at least one edge of $C^*$. Hence $|C^*| \geq |I^*|$.

(b) We need to show $|I^*| \geq |C^*|$. Let $I^*$ be a maximum independent set in $G = (A \cup B, E)$, and let $X := V(G) \setminus I^*$. The idea is to construct an edge cover $C$ of size $|I^*|$ by matching every vertex in $X$ with a vertex in $I^*$ and then add another edge to $C$ for every vertex in $I^*$ which was not covered by the matching. It then follows that for a minimum edge cover $C^*$ we have $|C^*| \leq |C| = |I^*|$.

The following construction is illustrated in Figure 1. Let $X_A := X \cap A, X_B := X \cap B$ and $I_A := I^* \cap A, I_B := I^* \cap B$. Consider the bipartite graphs $B_1 := G[I_A \cup X_A]$ and $B_2 := G[I_B \cup X_B]$. Observe that the above construction for an edge cover $C$ works out if $B_1$ contains a matching of size $|X_A|$ and $B_2$ contains a matching of size $|X_B|$.

Assume that $B_1$ does not contain a matching of size $|X_A|$. Then by Hall’s Theorem there exists $S \subseteq X_A$ such that $|S| > |\Gamma_B(S)|$. We set $I' := (I^* \setminus \Gamma_B(S)) \cup S$. Note that $I'$ is an independent set since $\Gamma_G(S) \cap I^* = \Gamma_B(S)$. Moreover, $|I'| > |I^*|$ which is a contradiction to the maximality of $I^*$. The same argument shows that $B_2$ contains a matching of size $|X_B|$.
Exercise 3 (Hall’s Theorem on Infinite Graphs)

We set \( A := \{a_0, a_1, a_2, \ldots \} \), \( B := \{b_1, b_2, \ldots \} \) and let \( G = (A \cup B, E) \) where \( E = \bigcup_{i \in \mathbb{N}} \{ \{a_0, b_i\}, \{a_i, b_i\} \} \) (see Figure 2). Then for every subset \( X \subseteq A \) we have \( |A| \leq |\Gamma(A)| \) but there is no matching in \( G \) that covers every vertex in \( A \).

Figure 1: Illustration of the construction in Exercise 2 (b).

Figure 2: Illustration of an infinite counterexample to Hall’s Theorem