Graphs and Algorithms

Edge Coloring I

By Vizing's theorem we know that $\chi'(G) \in \{\Delta, \Delta + 1\}$. In order to prove the statement it therefore suffices to show that $\chi'(G) \neq \Delta$. Suppose for the sake of contradiction that $G$ has a $\Delta$-edge coloring. As $G$ is $\Delta$-regular, among the edges incident at each vertex all colors appear exactly once. Hence the union of the edges in any two colors $i, j \in [\Delta]$ is a spanning subgraph $H(i, j)$ of $G$ that consists of even length cycles, each of them being colored alternatingly in one of the two colors.

(i) Consider any two colors $i, j \in [\Delta]$. As $H(i, j)$ consists of even length cycles the number of vertices of $H(i, j)$ and hence the number of vertices of $G$ is even, a contradiction.

(ii) Let $i$ be the color of a bridge $e$, and let $j$ be any other color. We know that $H(i, j)$ is a union of cycles, one of them containing $e$. But a bridge can never be an element of a cycle, hence we have a contradiction.

(iii) Let $v$ be an articulation vertex and $i, j$ the colors of two edges $e_i$ and $e_j$ that are incident to $v$ and that belong to different blocks. Again we know that $H(i, j)$ is a union of cycles, one of them containing both $e_i$ and $e_j$. But the only cycles an articulation vertex is contained in are cycles within one and the same block, hence we have a contradiction.

Edge Coloring II

It suffices to show that $G' := G - M$ has edge-chromatic number $\Delta := \Delta(G)$. We can then color $G$ with $\Delta + 1$ colors by coloring $G'$ with colors $1, 2, \ldots, \Delta$ and assigning color $\Delta + 1$ to all the edges of $M$.

Let $S$ be the set of vertices of degree $\Delta$ that are not covered by $M$. If $S = \emptyset$, $G'$ has maximum degree smaller than $\Delta$ and can therefore be colored with $\Delta$ colors by Vizing's theorem. If $S \neq \emptyset$, the vertices of $S$ form an independent set in $G$ (and in $G'$) because otherwise $M$ would not be maximal. Therefore, $G'$ is a graph with maximum degree $\Delta$ such that all the vertices of maximum degree form an independent set. We want to show that such a graph can always be edge-colored with $\Delta$ colors.

To color $G'$ with $\Delta$ colors, we can adapt the constructive proof of Vizing's theorem that was discussed in the lecture. First, we $\Delta$-color the subgraph of $G'$ that is induced by all vertices of degree less than $\Delta$. Note that this can be done because of Vizing's theorem. It now remains to color all edges that are incident to a vertex of degree $\Delta$. We color these edges in some arbitrary order. A new edge is assigned a color in $\{1, 2, \ldots, \Delta\}$ in exactly the same way a new edge is assigned a color in $\{1, 2, \ldots, \Delta + 1\}$ in Vizing's proof. Assume that $x$ is a vertex of degree $\Delta$ and that we want to color the edge $\{x, y\}$. For a vertex $u$, let $F_u$ be the set of colors that are missing at $u$, i.e. $F_u$ is the set of colors such that there is no edge with a color in $F_u$ incident to $u$. Note that $F_x \cap \{1, 2, \ldots, \Delta\} \neq \emptyset$ because at most $\Delta - 1$ edges incident to $x$ have been assigned a color before assigning a color to $\{x, y\}$. Note also that for all neighbors $z$ of $x$, $F_z \cap \{1, 2, \ldots, \Delta\} \neq \emptyset$ because $z$ has degree at most $\Delta - 1$. This is the only point where our proof differs from the proof of Vizing’s theorem. In general, we can only assume that for every neighbor $z$ of $x$, there is a free color in $\{1, 2, \ldots, \Delta + 1\}$ in $F_z$. The rest of the proof is identical to the proof of Vizing's theorem from the lecture.