Graphs and Algorithms

Exercise 1 (Greedy Coloring)

(a) Prove that there is an ordering \(v_1, v_2, \ldots, v_n\) of the vertices of \(G\) such that the greedy coloring algorithm yields an optimal coloring when using this ordering.

(b) Show that there is a tree \(T = (V, E)\) on \(n\) vertices and a permutation \(\pi : \{1, 2, \ldots, n\} \rightarrow V\) such that the algorithm \(\text{Greedy-Coloring}(T, \pi)\) needs \(\Omega(\log n)\) colors.

Exercise 2 (Completing the proof of Brooks’ Theorem)

In this exercise we will complete the proof of Brooks’ Theorem which was partly presented in the lecture.

Let \(G\) be 2-connected but not 3-connected. Assume that the vertices of every graph \(G'\) on at most \(n - 1\) vertices can be colored with \(\Delta(G')\) colors unless \(G'\) is a clique or an odd cycle in which case we need \(\Delta(G') + 1\) colors. Prove that under this assumption the vertices of \(G\) can be colored with \(\Delta(G)\) colors unless \(G\) is a clique or an odd cycle in which case we need \(\Delta(G) + 1\) colors.

Hints: First tackle the case \(\Delta(G) = 2\) and assume \(\Delta(G) \geq 3\) for the remainder. Now consider a vertex cut \(X := \{x, y\} \subseteq V\) of \(G\). Let \(C_1 \subseteq V\) be a connected component of \(G - X\). Let \(G_1 := G[C_1 \cup X]\) and \(G_2 := G[V \setminus C_1]\). First look at the case where \(\{x, y\}\) is an edge in \(G\). In the remaining case (\(\{x, y\}\) is not an edge in \(G\)) add the edge \(\{x, y\}\) to \(G\) and carefully apply the assumption.

Discussion of the exercises on 22.04.2010.