Exercise 1 - Ramsey numbers

(a) It was shown in the lecture that \( R(p, q) \leq R(p - 1, q) + R(p, q - 1) \) and also that \( R(3, 4) = 9 \) and \( R(k, 2) = k \) for every \( k \geq 2 \). As obviously \( R(k, 2) = R(2, k) \), it follows that \( R(3, 5) \leq R(2, 5) + R(3, 4) = 5 + 9 = 14 \).

(b) We need to show that every 3-coloring of \( E(K_{17}) \) contains a monochromatic triangle. A vertex \( v \in V(K_{17}) \) has 16 adjacent edges colored in three colors. Hence, we find at least \( \lceil \frac{16}{3} \rceil = 6 \) neighbors \( v_1, \ldots, v_6 \) of \( v \) such that the edges \( \{v, v_i\} \) for \( 1 \leq i \leq 6 \) all have the same color, say red. These six vertices span a \( K_6 \). If there is a red edge among the edges of this \( K_6 \), we have found a red triangle. Otherwise the edges of this \( K_6 \) are colored with only 2 colors. Because \( R(3, 3) = 6 \), this means that it has to contain a monochromatic triangle.

Exercise 2 - Outerplanar graphs

From a graph \( G = (V, E) \) construct a new graph \( G' = (V', E') \) by adding a new vertex and connecting it to all vertices of \( G \): \( V' = V \cup v', E' = E \cup \{\{v, v'\} | v \in V\} \).

Then \( G' \) is obviously planar if \( G \) is outerplanar. Furthermore, if \( G \) is planar we can delete \( v' \) from some drawing of \( G' \) in the plane to get a drawing of \( G \) in which all vertices are contained in the boundary of one face. Then we can choose a drawing of \( G \) with this face as the outer face. This shows that \( G \) is outerplanar if and only if \( G' \) is planar.

By Kuratowski, the planarity of \( G' \) is equivalent to \( G' \) not containing a subdivision of \( K_5 \) or of \( K_{3,3} \). Now, we can see that this is equivalent to \( G \) not containing a subdivision of \( K_4 \) or of \( K_{2,3} \). Indeed, if \( G' \) has a subgraph \( H \) isomorphic to a subdivision of \( K_5 \) or of \( K_{3,3} \) then either \( v' \notin V(H) \), i.e. \( H \) is also a subgraph of \( G \), or \( H \setminus \{v'\} \) contains a subdivision of \( K_4 \) or of \( K_{2,3} \). For the other direction, if \( G \) contains a subgraph \( H \) that is isomorphic to a subdivision of \( K_4 \) or of \( K_{2,3} \), \( H \) together with \( v' \) and the edges connecting \( v' \) to the vertices of \( H \) contain a subdivision of \( K_5 \) or of \( K_{3,3} \).