1 Trees

1. A $\delta$-regular tree is a tree where all inner vertices have degree $\delta$. Show that there is a $\delta$-regular tree with $n$ vertices if and only if $n \geq \delta + 1$ and $n \equiv 2 \pmod{\delta - 1}$.

2. Show that if $n \equiv 2 \pmod{\delta - 1}$, the number of labelled $\delta$-regular trees is $\binom{\sum_{i=1}^{\delta-1} \binom{n-1}{i}}{\binom{\delta-1}{\delta-1}}$ where the multinomial coefficient $\binom{m}{k_1, k_2, \ldots, k_t}$ is defined as $\prod_{i=1}^{t} \binom{m}{k_i}$.

2 Planar Graphs

1. Show that the complement of a simple planar graph with $n$ vertices is non-planar for $n \geq 11$.

2. Let $G$ be a planar graph with $n \geq 3$ vertices and $3n - 6$ edges embedded in the plane.

   (a) Show that all faces of $G$ are triangles.

   (b) Show that if $G$ has chromatic number 3, then it is Eulerian.

3 Connectivity

1. Let $G = (V, E)$ be a $k$-connected graph and let $D$ be the diameter of $G$. Show that $|V| \geq k(D - 1) + 2$ and that the size of the largest independent set of $G$ $\alpha(G) \geq \lceil (D + 1)/2 \rceil$.

2. Let $G = (V, E)$ be a graph with minimum degree $\delta(G) \geq |V|/2 + t$ for $0 \leq t < |V|/2 - 1$. Show that $G$ is $(2t + 2)$-connected.

3. Let $G$ be a graph with $n$ vertices such that any two distinct vertices $x$ and $y$ satisfy $\deg(x) + \deg(y) \geq n - 1$. Prove that $G$ is connected.

4. For each $n \geq 2$ give an example for a disconnected graph on $n$ vertices such that any two distinct vertices $x$ and $y$ satisfy $\deg(x) + \deg(y) \geq n - 2$.

4 Coloring

Let $k$ and $n$ be natural numbers with $k \geq 1$ and $n \geq k(k + 1)$. Place $n$ points on a circle and let $G_{n,k}$ be the 2$k$-regular graph obtained by joining each point to the $k$ nearest points in each direction on the circle. For example $G_{n,1}$ is a cycle on $n$ vertices.

1. Prove that $\chi(G_{n,k}) = k + 1$ if $k + 1$ divides $n$ and otherwise $\chi(G_{n,k}) = k + 2$.

2. Show that the lower bound on $n$ cannot be weakened by proving that $\chi(G_{k(k+1)-1,k}) > k + 2$ if $k \geq 2$. 