Topics in Random Graphs

This is the first out of four graded homework assignments. The best three grades out of those four count towards the final grade with a weight of 10% each. You are required to hand in your solutions by email at lgugelmann@inf.ethz.ch by 12:00 on April 15.

**Definition.** The graph $G = (V,E)$ is a $(k,t)$-expander if for all $V_0 \subseteq V$ with $|V_0| \leq k$ we have that $|N(V_0)| \geq t|V_0|$, i.e. that $V_0$ has at least $t|V_0|$ neighbors.

**Lemma.** Let $G$ be a $(k,2)$-expander. Then $G$ contains a path of length at least $3k-1$.

**Exercise 1 (20 Points)**

**Definition.** We write $G \rightarrow H$ if for all edge colorings of $G$ with two colors, $G$ contains a monochromatic copy of $H$.

**Definition.** Given a graph $H$ the size Ramsey number of $H$, denoted $\hat{r}(H)$ is the minimum number of edges in a graph $G$ such that $G \rightarrow H$.

**Example.** $K_6 \rightarrow K_3$, but $K_5 \not\rightarrow K_3$. $\hat{r}(K_3) \leq 15$.

The goal of this exercise is to prove the following statement:

**Theorem.** Let $P_k$ be a path on $k$ vertices. Then $\hat{r}(P_k) = O(k)$.

a) Prove that if a graph $G$ has average degree $d$, then it contains a subgraph $G_0$ with $\delta(G_0) \geq \frac{d}{2}$.

**Hint:** remove vertices with too small degree repeatedly. Argue that the remaining graph is not empty.

b) Let $C$ be some large enough constant and $G \sim G(n,C/n)$. Prove that with high probability for all subgraphs $G_0 \subseteq G$ with $\delta(G_0) \geq \frac{C}{5}$ we have that $G_0$ is a $(n/1000,2)$-expander.

**Hint:** assume that $G_0$ is not an expander. Then it contains a set $A$ of vertices which does not expand. Bound the number of edges in $A \cup N(A)$ and argue that whp $G$ does not contain such a dense subgraph.

c) Prove the theorem.

**Hint:** Consider a random graph $G(n,C/n)$ and assume that it has the right amount of edges and that it satisfies some condition related to b). Let an adversary color the graph and consider the majority color. Use parts a), b) and the lemma at the beginning of this exercise sheet.
Exercise 2 (10 Points)

Let $k \geq 1$ be an integer. Prove the following two statements:

a) With high probability a random graph $G(n, p)$ with

$$p \leq \frac{\log n + (k - 1) \log \log n - \omega(1)}{n}$$

has minimum degree $\delta(G(n, p)) \leq k - 1$.

b) With high probability a random graph $G(n, p)$ with

$$p \geq \frac{\log n + (k - 1) \log \log n + \omega(1)}{n}$$

satisfies $\delta(G(n, p)) \geq k$.

Hint: For part a) use the first and second moment method, for part b) a first moment calculation together with a union bound.

Exercise 3 (10 Points)

There are two independent parts to this exercise, you can choose which one you would like to prove. If you solve both, then the better solution of the two parts will determine the score for this exercise.

1) Let $\epsilon > 0$ and let $A, B$ be disjoint subsets of $[n]$ of size $|A| = |B| = \epsilon n$. Prove that there is a constant $C = C(\epsilon)$ such that a random graph $G(n, C/n)$ contains a path $P = (v_0, \ldots, v_\ell)$ of length $\ell \geq (1 - 3\epsilon)n$ which starts in $A$, ends in $B$ and has no other vertices in either set, i.e. $v_0 \in A$, $v_\ell \in B$ and $v_1, \ldots, v_{\ell-1} \in V \setminus (A \cup B)$.

Hint: expose a long path in $V \setminus (A \cup B)$ first, then prove that some vertices near the beginning and the end connect to $A$ and $B$ respectively.

2) For all $\epsilon > 0$ there is $r_0 > 0$ such that for every $r \geq r_0$ the random regular graph $G_{n,r}$ has with high probability a path of length at least $(1 - \epsilon)n$.

Hint: use the configuration model and the lemma at the beginning.