Topics in Random Graphs

Solution of Exercise 2

By Euler’s formula, we have $n - e + f \geq 2$ (with $n, e, f$ denoting the number of vertices, edges and faces of a planar graph). If a graph $G$ has girth at least $k$ then $f \leq \frac{2e}{k}$ and therefore $e \leq \frac{k}{k-2}(n-2) \leq \left(1 + \frac{2}{k-2}\right)n$.

We choose $k := \frac{4}{\epsilon} + 2$ (1)

Hence $\frac{2}{k} = \frac{2 \epsilon}{k-2}$ and therefore we have the following.

Claim 1. If a graph $G$ on $n$ vertices has girth at least $k$ then $|E(G)| \leq (1 + \frac{2 \epsilon}{k-2})n$.

Let $G = G(n, m)$ with $m \geq (1 + \epsilon)n$ and for every $i, 1 \leq i \leq n$, let $N_i$ and $N_{\leq i}$ denote the number of cycles of length $i$ and length at most $i$, respectively, in $G$. In the following, a cycle is called short if it has length at most $k$. The next claim bounds the expected number of short cycles.

Claim 2. We have $\mathbb{E}[N_{\leq k}] \leq c$ where $c$ does not depend on $n$.

Proof. Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$. For every sequence $i_1, i_2, \ldots, i_j$ with $j \leq k$ let $X_{i_1,i_2,\ldots,i_j}$ be the indicator variable for the event that the cycle $v_{i_1}, v_{i_2}, \ldots, v_{i_j}$ appears in $G$. So for every $j \leq k$,

\[
\mathbb{E}[N_j] = \sum_{i_1, i_2, \ldots, i_j} \mathbb{E}[X_{i_1,i_2,\ldots,i_j}]
\leq n^j \cdot \left(\frac{m}{\binom{n}{2}}\right)^j
\leq \left(\frac{m}{n-1}\right)^j 2^j
\leq (2 + 2\epsilon)^j \left(\frac{n}{n-1}\right)^j
\leq (2 + 2\epsilon)^j e^{\frac{\epsilon}{n-1}} \quad \text{(note that } \frac{n}{n-1} = 1 + \frac{1}{n-1})
\leq (2e + 2\epsilon)^j
\]

Hence $N_{\leq k} \leq k(2e + 2\epsilon)^k$. Setting $c := k(2e + 2\epsilon)^k$ proves the claim. \qed
Let $c$ be the constant from Claim 2. By Markov’s inequality we get $\mathbb{P}[N_{\leq k} \geq \log n] \leq \frac{\mathbb{E}[N_{\leq k}]}{\log n} \leq \frac{c}{\log n}$. So a.a.s. $G$ contains at most $\log n$ short cycles. We remove one edge per short cycle and let $G'$ denote the resulting graph. Hence a.a.s. $G'$ contains at least $m - \log n \geq (1 + \epsilon)n - \log n \geq (1 + \frac{3}{4})n$ edges. Note that the girth of $G'$ is at least $k$. If $G'$ is planar then by Claim 3 $|E(G')| \leq (1 + \frac{3}{4})n$. But a.a.s. $G'$ has more edges and thus cannot be planar. This means that a.a.s $G$ is non-planar as well.

Solution of Exercise 3

Let $n = t^{1+\epsilon}$. We choose $p$ such that

$$\frac{4(1 + \epsilon) \ln t}{t - 1} < p < \frac{1}{2} t^{-\frac{2(1+\epsilon)}{3}}. \tag{2}$$

This is possible since for $\epsilon$ small and $t$ large enough, $\frac{4(1 + \epsilon) \ln t}{t - 1} \ll \frac{1}{2} t^{-\frac{2(1+\epsilon)}{3}}$.

Let $G = G(n,p)$. For an integer $k$ let $C_k$ denote the number of cliques of order $k$ and let $I_k$ denote the number of independent sets of order $k$. We have $\mathbb{E}[C_k] = {n \choose k} p^{\frac{k(k+1)}{2}}$ and $\mathbb{E}[I_k] = {n \choose k} (1 - p)^{\frac{k(k+1)}{2}}$.

The goal is to show that $\mathbb{E}[C_4] < \frac{1}{2}$ and $\mathbb{E}[I_t] < \frac{1}{2}$. This directly implies that $\mathbb{E}[C_4 + I_t] < 1$ and thus guarantees that $R(4, t) > n$.

By (2) we have $\mathbb{E}[C_4] = {n \choose 4} p^6 \leq t^{4(1+\epsilon)} p^6 < \frac{1}{2}$. Moreover,

$$\mathbb{E}[I_t] = {n \choose t} (1 - p)^{\frac{t(t+1)}{2}} \leq n^t e^{-p \frac{t(t+1)}{2}} \leq n^t e^{-pt(1-\epsilon)} = \left( n^{1+\epsilon} e^{-pt(1-\epsilon)} \right)^t \leq \frac{1}{2}$$

The last inequality follows from (2). Hence we are done.

Solution of Exercise 4

We only prove the statement for the clique number, the one for the independence number is identical as the probability of having an edge or non-edge are equal.

Let $X$ denote the number of cliques of size $k$ in $G(n, 1/2)$. It is easy to see that:

$$\mathbb{E}[X] = {n \choose k} 2^{-\frac{k(k+1)}{2}}$$

We want to prove sharp concentration of $X$ around this value, so we need to bound the variance of $X$. We have that

$$\mathbb{E}[X^2] = \sum_{i=0}^{k} {n \choose k} {k \choose i} (n-k)^{i} 2^{-2\frac{k(k+1)}{2} + \frac{i(i+1)}{2}}.$$
The sum runs over \( i \), the number of vertices in which two cliques of size \( k \) can overlap. The binomial coefficients represent the choices for the vertices in the first clique, the choice of the \( i \) overlap-vertices and the choices for the remaining vertices of the second clique respectively.

The remaining term of the variance is given by

\[
E[X]^2 = \sum_{i=0}^{k} \binom{n}{k} \binom{k}{i} \binom{n-k}{k-i} 2^{-2\binom{i}{2}}.
\]

The sum results from applying Vandermonde’s identity.

Combining the last two equations we have that:

\[
\frac{\text{Var}(X)}{E[X]^2} = \frac{\sum_{i=2}^{k} \binom{n}{k} \binom{k}{i} \binom{n-k}{k-i} 2^{-2\binom{i}{2}} (2\binom{2}{2} - 1) = \sum_{i=2}^{k} \binom{k}{i} \frac{(n-k)(2\binom{2}{2} - 1)}{\binom{n}{k}}.}
\]

With some effort it is possible to see that the above equation is indeed \( o(1) \), so by Chebyshev’s inequality \( X \sim E[X] \) for \( n \to \infty \).

Now set \( k = \lceil 2 \log_2 n \rceil \). Then

\[
E[X] = \left( \frac{n}{k} \right)^{2^{-\binom{2}{2}}} \leq \left( \frac{en}{k} \right)^{2^{-\binom{2}{2}}} + \frac{1}{2} = \left( \frac{en^{2-k/2}2^{2}}{2} \sqrt{\frac{2}{k}} \right)^{k} \leq \left( \frac{e2^{2}}{2} \right)^{k} = o(1).
\]

On the other hand we have for e.g. \( k = \lceil 2 \log_2 n - 2 \log_2 2 \log_2 n \rceil \) that

\[
E[X] \geq \left( \frac{n}{k} \right)^{2^{-k/2+k/2}} = \left( \frac{en^{2-k/2}2^{2}}{2} \sqrt{\frac{2}{k}} \right)^{k} \geq \left( \frac{2\sqrt{2} \log_2 n}{2 \log_2 n - 2 \log_2 2 \log_2 n} \right)^{k} \geq \sqrt{2} \to \infty
\]

**Solution of Exercise 5**

From Exercise 4 we know that for \( G \sim G(n, 1/2) \) we have independence number \( \alpha(G) = (1 + o(1))2 \log_2 n \). By using the inequality \( \chi(G) \geq \frac{\log_2 |G|}{\log_2 \alpha(G)} \) we immediately have that \( \chi(G) > C_1 \frac{n}{\log_2 n} \) for some constant \( C_1 \).

**Proposition 3.** For some constant \( C_2 \) with high probability \( G \) contains no subgraph \( H \) on \( k > C_2 \log_2 n \) vertices such that \( H \) contains more than \( \frac{3}{4} \binom{k}{2} \) edges.

**Proof.** The number of edges in a fixed subgraph \( H \) is distributed according to \( \text{Bin}(k, 1/2) \).

Using Chernoff-type bounds we know that \( \mathbb{P}[E(H) > \frac{3}{4} \binom{k}{2}] < e^{-c_3 k^2} \). With a union bound we obtain that

\[
\mathbb{P}[\exists H \subset G \text{ s.t. } \ldots] \leq \binom{n}{k}e^{-c_3 k^2} = o(1).
\]

We now want to bound the Hajos number of \( G \). Assume that \( \text{Haj}(G) \geq k > C_2 \log_2 n \). Then we can find a set \( S \subset V(G) \) of \( k \) vertices such that there exists a subdivision of \( K_k \) on these vertices. Denote with \( H \) the subgraph of \( G \) induced by \( S \). From the proposition above we know that \( H \) is missing at least \( \frac{1}{4} \binom{k}{2} \) edges. To obtain a subdivision of \( K_k \) on \( S \) each missing edge must be compensated for by a path of length \( \geq 2 \) with at least one vertex outside of \( H \). As all these paths must be non-intersecting we have that there can be at most as many as there are vertices in \( G \setminus H \): \( \frac{1}{4} \binom{k}{2} < n - k \). The Hajos number for \( G \) is therefore bounded from above by \( k \leq C_3 \sqrt{n} \), which for \( n \) large enough is smaller than \( C_1 \frac{n}{\log_2 n} \).