Graphs and Algorithms

Exercise 1

Recall that for a square matrix $A = (a_{ij})_{i,j \in [n]}$, we define the trace of $A$ as the sum of its diagonal entries, i.e.,

$$\text{tr } A = \sum_{i=0}^{n} a_{ii}.$$ 

Let $G$ be a graph and $A_G$ its adjacency matrix.

(i) What does $\text{tr}(A_G^2)$ mean in terms of the graph?

(ii) Describe a method to determine the number of triangles—i.e., the number of subgraphs isomorphic to a triangle—in $G$ using matrix multiplication. Prove correctness.

(iii) Similarly describe a method to determine the number of cycles of length 4 in $G$, and prove it correct.

Exercise 2

Let $G$ be a graph. Prove that $G$ is Hamiltonian if for all non-adjacent vertices $u, v$ we have $\text{deg } u + \text{deg } v \geq n$.

Note: an earlier online version said “iff”, which would imply $P = NP$.

Exercise 3

Let $G$ be a plane graph and $G^*$ its dual graph. Prove that a cycle in $G$ is a minimal edge cut in $G^*$, and vice-versa.

Exercise 4

A plane graph is a triangulation if every face (including the outer face) has exactly three adjacent edges.

(a) Prove that every triangulation on at least 13 vertices has at least one vertex of degree at least 6.

Hint: Assume all vertices have degree at most 5 and bound the number of edges.

(b) Find a planar graph on six vertices, where all vertices have degree 4; or prove that no such graph can exist.

(c) Find a planar graph on six vertices and 13 edges, or prove that no such graph can exist.

(d) Let $G$ be a connected planar graph on $n \geq 1$ vertices, where all vertices have degree at least 4. Show that in any embedding of $G$ in the plane there are at least $n + 2$ faces.

Voluntary solutions may be handed in until 21.04.2010.