**Exercise 1**

Consider a two-player path-building game between Alice and Bob, played on a graph $G$. Alice picks a starting vertex $v$. They then alternate (starting with Bob) in picking a neighbor of the last vertex. Every vertex may only be chosen once. In this way, they build a path in $G$ starting at $v$. The first player who cannot make a legal move (because there are no more vertices, or because there are no unvisited neighbors) loses.

Prove that Bob wins if and only if $G$ has a perfect matching. \(8 \text{ Points}\)

*Hint:* Alice should fix a maximal matching, and start with an unmatched vertex.

**Exercise 2**

A permutation matrix is a matrix $P = (p_{ij})$ with entries $p_{ij} \in \{0, 1\}$ such that in every row and every column there is exactly one nonzero entry.

Let $A = (a_{ij})$ be an $n \times n$ matrix with entries $a_{ij} \in \mathbb{N}$. Prove that $A$ can be decomposed into $k$ permutation matrices $P_1, \ldots, P_k$ as

\[ A = \sum_{i=0}^{k} P_i \]

if and only if the row and column sums of $A$ all equal $k$, i.e., if

\[ \sum_{i=0}^{n} a_{i\ell} = \sum_{j=0}^{n} a_{\ell j} = k \]

for all $\ell$. \(8 \text{ Points}\)

*Hint:* Given a subset $R \subseteq [n]$ of the rows, look at the sum $\sum_{j \in [n]} \sum_{i \in R} a_{ij}$. (As usual, $[n] = \{1, \ldots, n\}$.)
Exercise 3
Let $G$ be an $n$-edge-connected graph, and $L(G)$ its line graph. Prove that:

(i) $L(G)$ is $n$-vertex-connected. (4 Points)

(ii) $L(G)$ is $(2n - 2)$-edge-connected. (4 Points)

*Hint:* What does an edge cut in $L(G)$ mean when transferred back to $G$?

Exercise 4
Let $G = (V, E)$ be a graph. For $u, v \in V$ we define $\kappa(u, v)$ as the number of internally vertex-disjoint $u$-$v$-paths in $G$. We thus have

$$\kappa(G) = \min_{u, v \in V} \kappa(u, v).$$

We call $G$ minimally $n$-(vertex-)connected if it is $n$-connected and for any $e \in E$, the graph $G - e$ is at most $(n - 1)$-connected.

Prove:

(i) If $G$ is minimally $n$-connected, then it is not $n + 1$-connected. (4 Points)

(ii) $G$ is minimally $n$-connected if and only if $\kappa(u, v) = \kappa(G)$ for all $(u, v) \in E$. (4 Points)

Submit your solutions by email until 31.03.2011.