Let \( \alpha'(G) \) be the maximum cardinality of a matching of \( G \), and let \( \beta'(G) \) be the minimum cardinality of an edge cover of \( G \).

**Exercise 1**

Let \( G = (V, E) \) be a graph with no isolated vertices. Prove that \( \alpha'(G) + \beta'(G) = \lvert V \rvert \).

**Exercise 2**

Let \( G = (A \uplus B, E) \) be a bipartite graph. Prove that

\[
\alpha'(G) = \lvert A \rvert - \max_{S \subseteq A} (\lvert S \rvert - \lvert \Gamma(S) \rvert).
\]

**Exercise 3**

Let \( G = (V, E) \) be a graph. We define its total graph \( T(G) \) by taking as vertices \( V \cup E \), and defining the adjacency relation (i.e., the edge set) via

(i) \( v \in V \) and \( u \in V \) are adjacent in \( T(G) \) if they are neighbors in \( G \).

(ii) \( v \in V \) and \( e \in E \) are adjacent in \( T(G) \) if \( v \in e \) (that is, \( v \) is an endpoint of \( e \) in \( G \)).

(iii) \( e \in E \) and \( f \in E \) are adjacent in \( T(G) \) if they are coincident in \( G \) (that is, they share an endpoint).

Prove that if \( G \) is connected and has at least two vertices, then \( T(T(G)) \) is Hamiltonian.

**Hint:** First prove that \( T(G) \) contains a spanning subgraph which is Eulerian.

**Exercise 4**

a) What is the minimum connectivity of a Hamiltonian graph on \( n \) vertices?

b) What is the maximum connectivity of a non-Hamiltonian graph on \( n \) vertices?

Submit your solutions by email until 14.04.2011.