Graphs and Algorithms

For lack of time, these are only short proof sketches.

Proof Sketch for Exercise 1
(a) $K_4$.
(b) W.l.o.g. assume no line is parallel to y-axis (otherwise wiggle a bit). Sort by ascending $x$ coordinates, color greedily. Then by the time you see a vertex, you have only colored two of its neighbors.
(c) Cute solution: Add vertex $v$ in $F$, connect it to all existing vertices. Use four-color theorem. Remove $v$.

Originally intended solution: Assume $G$ biconnected. Either $F$ is a cycle and you need only three colors; or there is an edge $e$ “inside” the cycle, split along $e$ in two halves and use induction. Take care to fix colors when assembling at $e$. Similarly assemble 3-colored biconnected components into an arbitrary 3-colored graph.

Proof Sketch for Exercise 2
(a) Greedily color in the given order. Both $i$ and $\deg v_i + 1$ are simple bounds on how many neighbors of $v_i$ can already have received a color.
(b) Look at color classes in an optimal coloring. Between any pair of classes there must be at least one edge. Solve for $\chi(G)$.

(The hint relates to our originally intended solution, which involves ordering the vertices by decreasing degrees.)

Proof Sketch for Exercise 3
(a) Biggest color class in $G$ has at least $n/\chi(G)$ members. It is a clique in $\bar{G}$, so $\chi(\bar{G}) \geq n/\chi(G)$.
(b) For lower bound, use arithmetic-geometric mean bound

$$\sqrt{xy} \leq \frac{x + y}{2} \quad \text{for } x, y \geq 0$$

with (a).

For upper bound, use induction. Remove a vertex $v$ and look at how the $\chi$’s change. If the result violates the claimed bound, look at the number of neighbors of $v$ instead.

Proof Sketch for Exercise 4
1. Suppose not, then $d(x,y)$ along any path or the fixed cycle is odd/even. Split the $y$’s into two classes according to the parity of their distance from $x$. This gives a bipartition.

2. All of $\Gamma(v)$ must be on $P$ otherwise it was not maximal. Show that any $w \in \Gamma(v)$ must be at an odd distance from $v$ as otherwise you can construct a long odd cycle.

3. Remove vertices with degree $< 2k$ as long as possible; those can be colored greedily later. Look at the resulting subgraph where $\delta \geq 2k$. It is non-empty as otherwise $\chi \leq 2k$. Use (b) to get a very long even cycle and then (a) to construct a long odd cycle, contrary to the assumptions.