Graphs and Algorithms

For lack of time, these are only short proof sketches.

Proof Sketch for Exercise 1

(i) Number of walks of length 2.

(ii) Similar to (i), \( \text{tr} A^3_G \) is the number of walks of length 3. Divide by 6 to account for choice of starting point and starting direction.

(iii) Similar to (ii) but now you also have to subtract walks which are not cycles of length 4, such as \((u, v, u, v, u), (u, v, w, v, u)\) etc. Again account for symmetry and starting point.

Proof Sketch for Exercise 2

Assume you have a maximal (in the number of edges) counterexample. It must contain a Hamiltonian path. Use the precondition to find a neighbor common to start and end of the path. Use it to construct a Hamilton cycle.

Proof Sketch for Exercise 3

W.l.o.g. let \( G \) be connected. Fix a cycle \( C \). In \( G^* \), faces inside of \( C \) are no longer connected to faces outside, giving a cut. Attempting to reinsert an edge will reconnect the components.

Conversely, fix a minimal edge cut in \( G^* \). There are only two components by minimality. Unless the cut has only one edge each face is adjacent to two cut edges, yielding a closed walk. Any setting in which the walk is not a cycle contradicts minimality of the cut.

Proof Sketch for Exercise 4

(a) \( e = 3n - 6 \) and \( 2e \leq 5n \) together imply \( n \leq 12 \).

(b) Octahedron (see GHW4 solutions for a drawing).

(c) \( e \leq 3n - 6 = 12 \).

(d) Degree bound gives \( e \geq 2v \). Substitute into Euler formula \( v + f - e = 2 \).