Graphs and Algorithms

Exercise 1 – Matchings in Trees

Prove that every tree admits at most one perfect matching.

Exercise 2 – Stable Matchings

There are $n$ men and $n$ women in a small mountain village, where each person has ranked all members of the opposite sex in order of preference. They all wish to marry a person they rank as high as possible in their list. The local priest tries to play matchmaker such that all marriages are stable. His goal is to have no two people which prefer each other to their current spouse. (Otherwise they would leave their current partner and marry each other.)

Consider the following algorithm he devised to find such a stable matching. In each round each man proposes to the highest woman on his preference list who has not previously rejected him. If each woman receives exactly one proposal, stop and use the resulting matching. Otherwise, every woman receiving more than one proposal rejects all of them except the one that is highest on her preference list. Every woman receiving a proposal says “maybe” to the most attractive proposal received.

a) Prove that no man can be rejected by all the women.

b) Prove that the algorithm terminates.

c) Prove that at termination the algorithm finds a stable matching.

Exercise 3 – Knights on a Chessboard

What is the maximum number of knights that can be placed on a chessboard such that no two threaten each other?

Hint: Formulate the problem in the language of graph theory and exploit the relations between independent sets, vertex covers and matchings. Fun fact: you can start with a knight on any square and jump around visiting each square exactly once.

Discussion of the solution in the exercise class on 5.4.2012.