Graphs and Algorithms

Exercise 1

Let $G = (V,E)$ be a connected graph and let $B(G) = (A \cup B,E')$ denote its block graph, where $A \subset V$ is the set of articulation points, $B$ the set of blocks and $E' = \{ \{a,b\} \mid a \in A, b \in B, a \in V(b) \}$.

We need to show that $B(G)$ is a tree, i.e. that it is a connected and acyclic graph.

First we review some facts which follow easily from the definition of a block graph.

**Fact 1.** The intersection of two blocks consists of at most one vertex and this vertex is an articulation point.

**Proof.** The statement is true for blocks which do not intersect. Assume now that two blocks $b_1,b_2$ intersect in at least two vertices $v$ and $w$. There cannot be an edge between $v$ and $w$ as this would contradict the maximality of both $b_1$ and $b_2$. We can find in both $b_1$ and $b_2$ a path from $v$ to $w$ using only edges of $b_1$ resp. $b_2$ with the following procedure. Take any edge incident to $v$ and any edge incident to $w$ in $b_1$ or $b_2$. By definition these two edges must be part of a cycle containing only edges of $b_1$ resp. $b_2$. In particular this implies that there is a path from $v$ to $w$. Now join the $v$-$w$ path in $b_1$ with the one in $b_2$. This is a cycle and contradicts the maximality of $b_1$ and $b_2$ as maximal biconnected components.

It remains to show that if the intersection consists of exactly one vertex $v$, then this vertex is an articulation point. Let $w_1$ and $w_2$ be vertices of $b_1,b_2$ adjacent to $v$. If $v$ is not an articulation point, then $G \setminus v$ is connected and contains a path from $w_1$ to $w_2$ not using $v$. We can turn this path into a cycle adding the edges $\{w_1,v\}$ and $\{v,w_2\}$. However this cycle uses edges of both $b_1$ and $b_2$, again a contradiction to their maximality as biconnected components.

**Fact 2.** Each edge is contained in exactly one block.

**Proof.** Assume that at least one edge is contained in no block. But then by definition it is a block itself. If one edge is contained in two blocks then their intersection contains at least two vertices, which contradicts the previous fact.

**Claim 1.** $B(G)$ is connected.

**Proof.** Assume that this is not the case. Then there are two vertices $x$, $y$ of $B(G)$ which are not in the same component. Take any vertex $v \in x$ and $w \in y$ in $G$. As $G$ is connected there is a path $P$ from $v$ to $w$. This path consists of edges and by Fact 2 we can map each edge uniquely to a block. Therefore we can segment $P$ in smaller paths which are completely contained in one block. The intersection between subsequent path-segments must be an articulation point by Fact 1. But this means that we can map $P$ to a path $P'$ in $B(G)$ by simply taking the alternating $B - A$ path that starts with the block corresponding to the first path segment, goes to the intersection of the first and second path segment, then the block corresponding to the second segment and so on. Note that $P'$ must not yet necessarily start on $x$ and end on $y$. This is the case if $v$ or $w$ are articulation
points. But then we can trivially extend \( P' \) to include \( x \) and \( y \) by either adding them directly to the path (if they are articulation points) or by adding the articulation point between \( v \) and the end of the path (i.e. \( x \)) and \( v \) itself (and equivalently for \( w \)).

**Claim 2.** \( B(G) \) is cycle free.

*Proof.* Assume that \( B(G) \) contains a cycle \( C \). As \( B(G) \) is bipartite this cycle must have length at least 4 and contain at least two blocks \( b \) and \( b_1 \). Remove \( b \) from \( C \) and denote with \( P' = (a_0,b_1,\ldots,a_n) \) the resulting path. Each 3 consecutive vertices of \( P' \) of the form \( (a_1,b,a_2) \) where \( a_1,a_2 \in A \) and \( b \) in \( B \) can be mapped to a path from \( a_1 \) to \( a_2 \) in \( G \) only using edges of \( b \). By concatenating all these paths we obtain a path \( P \) in \( G \) from \( a_0 \) to \( a_n \) which does not use any edges of \( b \). Note that \( P' \) must contain at least one block \( b_1 \neq b \) and therefore \( P \) must contain some edges which are not part of \( b \). The union of \( b \) and \( P \) is now a biconnected component larger than \( b \), contradicting the maximality of \( b \). \( \square \)

**Exercise 2 (Solution by Hörður Freyr Yngvason)**

Fully characterize the set of graphs such that a DFS and BFS ordering of their vertices are equal.

*Solution.* Without loss of generality, we assume all graphs to be connected. We want to characterize the enumerations produced by DFS and BFS using a common set of traits and then unify these traits such that the enumerations must agree. More precisely, note that the ordering produced is simply a permutation \( \sigma : V \rightarrow V \), and if we identify \( V \equiv \{1,\ldots,n\} \) then \( \sigma \in S_n \). So we define

\[
\text{DFS}_n := \{ (G, \sigma) : \sigma \text{ is a DFS-ordering on } G \}
\]

and define \( \text{BFS}_n \) analogously. Then, the set we are interested in is \( \text{BFS}_n \cap \text{DFS}_n \). We write \( x <_{\sigma} y \) to denote \( \sigma(x) < \sigma(y) \). First, observe that any \( \sigma \) generated by DFS or BFS always has \( \sigma(1) = 1 \), because we assume BFS and DFS start from vertex 1.

Now, consider \( \sigma \) generated by BFS. Let \( x,y,z \in V \) such that \( x <_{\sigma} y <_{\sigma} z \) and \( \{x,z\} \in E \). Notice that when the BFS reaches \( z \) from \( x \), then \( y \) must already have been visited from some vertex \( w \in V \), such that \( \{w,y\} \in E \) and \( w <_{\sigma} x \). It turns out that this characterizes BFS orderings on \( n \) vertices completely.

**Lemma.** \( (G, \sigma) \in \text{BFS}_n \) if and only if \( \sigma(1) = 1 \) and for all \( x,y,z \in V \) such that \( x <_{\sigma} y <_{\sigma} z \) and \( \{x,z\} \in E \) there exists \( w \in V \) such that \( \{w,y\} \in E \) and \( w <_{\sigma} x \).

*Proof.* We’ve already shown the “if”-part, but we still need to show that if \( \sigma \) has the above properties on some \( G \), then \( \sigma \) is a BFS-ordering on \( G \). Now, clearly, \( \sigma \) is a BFS-ordering on \( G \) if and only if the BFS-ordering on \( \sigma[G] \) is the identity mapping \( x \mapsto x \) (where \( \sigma[G] \) is the relabeling of \( G \) according to \( \sigma \)). So assume we have \( G = (V,E) \) with \( V = \{1,\ldots,n\} \), such that for all \( x < y < z \) with \( \{x,z\} \in E \), there exists \( w \in V \) such that \( w \leq x \) and \( \{w,y\} \in E \). We want to show that BFS visits the vertices of \( G \) in the order \( 1,\ldots,n \).

By definition, BFS starts from vertex number 1. By connectedness, there exists \( z \) such that \( \{1,z\} \in E \). If \( z \neq 2 \), then we have \( 1 < 2 < z \) so by the condition, we have \( \{1,2\} \in E \). Hence, the first vertex visited from 1 is vertex 2. Next, assume the vertices \( 1,\ldots,k \) have already been visited in the correct BFS order. We need to show that the next vertex visited will be \( k+1 \). In other words, we need to show that \( k+1 \) has a parent in \( \{1,\ldots,k\} \) that is no larger than the parent of any \( z \in \{k+2,\ldots,n\} \). So assume we have \( x < k+1 < z \) with \( \{x,z\} \in E \). Then there exists \( w \leq x \) such that \( \{w,k+1\} \in E \). Hence, \( k+1 \) is the vertex that will be visited next. By induction, we get that the vertices \( 1,\ldots,n \) are already in a BFS order. \( \square \)
Next, consider \( \sigma \) generated by DFS. Let \( x, y, z \in V \) such that \( x \prec_\sigma y \prec_\sigma z \) and \( \{x, z\} \in E \). Then there must be a path from \( x \) to \( y \) used by the DFS, containing only vertices \( v \) such that \( x \preceq_\sigma v \preceq_\sigma y \).

**Lemma.** \((G, \sigma) \in DFS_n\) if and only if \( \sigma(1) = 1 \) and for all \( x, y, z \in V \) such that \( x \prec_\sigma y \prec_\sigma z \) and \( \{x, z\} \in E \), there is a path from \( x \) to \( y \) containing only vertices \( v \) such that \( x \preceq_\sigma v \preceq_\sigma y \).

**Proof.** Like for BFS, we only need to show that given a relabeled graph with such an ordering, DFS visits the vertices in the order 1, \ldots, \( n \). So assume that for all \( x < y < z \) with \( \{x, z\} \in E \) that there is a path from \( x \) to \( y \) using only vertices in \( \{x, x+1, \ldots, y\} \).

By definition, DFS starts from 1. By connectedness, there exists \( \{1, z\} \in E \) with \( z \geq 2 \), and if \( z > 2 \) then we have \( 1 < 2 < z \), so there is a path from 1 to 2 using only the vertices in \( \{1, 2\} \); in other words, \( \{1, 2\} \in E \). Now, assume DFS has visited the vertices \( \{1, \ldots, k\} \) in their numerical order. We want to show that then \( k+1 \) will be the next visited vertex. In other words, we want to show that if \( x \) is the largest vertex in \( \{1, \ldots, k\} \) that still has an unvisited child \( z > k+1 \), then \( k+1 \) is also a child of \( x \). For such \( x, z \) we have \( x < k+1 < z \) and \( \{x, z\} \) is an edge. By the DFS-condition there exists a path from \( x \) to \( k+1 \) using only vertices in \( \{x, x+1, \ldots, k+1\} \), so \( k+1 \) is a child of some \( y \in \{x, \ldots, k\} \), so \( k+1 \) is a child of \( x \). Hence, \( k+1 \) is the vertex that DFS visits next. By induction, we get that the vertices 1, \ldots, \( n \) are already in a DFS order.  

The complete characterization follows immediately from the previous two lemmas:

**Corollary.** \( DFS_n \cap BFS_n \) is the set of pairs \((G, \sigma)\) such that \( \sigma(1) = 1 \) and for all \( x \prec_\sigma y \prec_\sigma z \) with \( \{x, z\} \in E \), there exist \( w \in V \) such that \( \{w, y\} \in E \), and \( w \prec_\sigma x \), and there is a path from \( x \) to \( y \) using only vertices \( v \) such that \( x \preceq_\sigma v \preceq_\sigma y \).

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**Exercise 3**

a) Assume that \( v \) is a vertex of maximal out-degree, but not a king. Then there must be a vertex \( w \) at distance at least 3 from \( v \). This implies that there must be an arc from \( w \) to \( v \) (otherwise \( w \) would be at distance 1) and an arc from \( w \) to all vertices of \( \Gamma^+(v) \) (otherwise \( w \) would be at distance 2 from \( v \)). But now we have that \( |\Gamma^+(w)| \geq |\Gamma^+(v)| + 1 \), contradicting the fact that \( v \) has maximal out-degree.

b) Let \( v \) be an arbitrary vertex in \( D \). Note that \( V(D) = \{v\} \cup \Gamma^+(v) \cup \Gamma^-(v) \). Consider the graph induced by \( \Gamma^-(v) \) in \( D \). It is also a tournament, and by a) it must contain at least one king \( w \) (any vertex with maximal out-degree will do). The vertex \( w \) is therefore at distance at most 2 to any vertex in \( \Gamma^-(v) \). But \( w \in \Gamma^-(v) \) it is at distance 1 to \( v \) and at distance at most 2 to \( \Gamma^+(v) \), as these vertices are at distance 1 to \( v \). By the note above it must therefore be a king in the entire \( D \). As the above holds for arbitrary \( v \) it must in particular also hold if \( v \) is a king.

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**Exercise 4**

a) We need to prove that any graph \( G \) with \( \delta(G) \geq k \) contains all trees with at most \( k \) edges.

We prove this by induction on \( k \). Let \( k = 1 \), then \( \delta(G) \geq 1 \) and therefore \( G \) contains at least one edge and we can trivially embed the unique tree with 1 edge.

Consider now a graph \( G \) with \( \delta(G) \geq k+1 \). By induction we can embed all trees with \( k \) edges in \( G \) \( \delta(G) \geq k+1 \) in particular implies \( \delta(G) \geq k \). Let \( T \) be any tree with \( k+1 \) edges and let \( e \) denote a leaf of \( T \). We also denote with \( v_e \) the endpoint of \( e \) which has not degree 1.
(this always exists, as $T$ is connected and has at least two edges). By induction we can embed $T' = T \setminus e$ into $G$. Let $w_e$ denote the vertex of $G$ corresponding to $v_e$ in the embedding of $T'$. If $w_e$ has a neighbor in $G$ which is not part of the embedding of $T'$, then we can identify $e$ with the edge from $w_e$ to this neighbor and are done. This is however always the case: $T'$ has exactly $k + 1$ vertices, i.e. $w_e$ can have at most $k$ neighbors that are part of the embedding of $T'$. As $\delta(G) \geq k + 1$ there must be at least one neighbor of $w_e$ not part of the embedding of $T'$.

b) We need to prove that if a graph $G = (V,E)$ with $n \geq k + 1$ vertices satisfies $|E| > n(k-1) - \binom{k}{2}$, then we can embed every tree with at most $k$ edges into it.

First note that it is sufficient to prove the statement for all trees with exactly $k$ edges, as we can take any tree with $\ell < k$ edges, add a path of length $k - \ell$ to any of its vertices, embed this new tree with $k$ edges and then erase the path in the embedding.

Let $k$ be arbitrary but fixed. Note that for $n = k + 1$ (the minimal value for $n$) we have that

$$|E| > n(k-1) - \binom{k}{2} = (k + 1)(k - 1) - \frac{k(k-1)}{2} = \frac{k(k-1)}{2} + k - 1 = \binom{k + 1}{2} - 1.$$ 

In other words $E = \binom{k+1}{2}$ and the graph $G$ must be the complete graph on $k + 1$ vertices. But then we know that $\delta(G) \geq k$ and by part a) we are done. We take this as the base case of an induction on $n$.

So assume that we have a graph $G$ on $n + 1$ vertices satisfying the condition of the theorem. If $\delta(G) \geq k$ we are done by a), so assume this does not hold. Then $G$ contains at least one vertex $v$ with degree at most $k - 1$. Let $G'$ denote the graph $G$ with $v$ removed. We have

$$|E(G')| \geq |E(G)| - (k - 1) > (n + 1)(k-1) - \binom{k}{2} - (k - 1) = n(k-1) - \binom{k}{2}.$$ 

Note that $G'$ satisfies the conditions of the theorem for a graph on $n$ vertices, so by induction we can embed all trees of at most $k$ edges in the subgraph $G'$ of $G$, and therefore in $G$ as well.