Graphs and Algorithms

Exercise 1 (Connectivity)

Let $G$ be a graph. Prove the following statements.

(a) Let $s, t$ be two different vertices of $G$. If there is a walk starting in $s$ and ending in $t$ then there is a path with leaves $s$ and $t$.

(b) We define $s \sim t : \iff$ there exists a path from $s$ to $t$.

Then “$\sim$” is an equivalence relation, i.e., it is reflexive, symmetric and transitive.

(c) $G$ is the disjoint union of its connected components.

Exercise 2 (Properties of Trees)

Prove the following statements:

(a) Let $T$ be a tree.

(i) If $n := |V(T)| \geq 2$ then $T$ contains at least two leaves.

(ii) Deleting a leaf from $T$ produces another tree.

(b) (Characterization of trees) For a graph $G$ on $n$ vertices, the following are equivalent:

- $G$ is connected and has no cycles.
- $G$ is connected and has $n - 1$ edges.
- $G$ has $n - 1$ edges and no cycles.
- For each $u, v \in V(G)$, $G$ has exactly one $u, v$-path.

(c) Every edge of a tree is a bridge.

(d) Adding one edge (and no vertices) to a tree forms exactly one cycle.

(e) Every connected graph contains a spanning tree.

Exercise 3 (Bridge-It)

(a) In a Bridge-it game, show that when no more moves are possible then exactly one player has built a bridge.

(b) Describe an explicit winning strategy for player 1. (This includes proving that your strategy is successful.)
Exercise 4 (Strategy Stealing)

In the lecture it was shown by a strategy stealing argument that the player who makes the first move wins Bridge-it.

Claim. The player who makes the second move wins Bridge-it.

Proof (strategy stealing). Assume for the sake of contradiction that Player 1 has a winning strategy. After Player 1 made his first move, Player 2 ignores this move and pretends to be Player 1 by stealing his winning strategy. Hence, Player 2 wins the game, which contradicts our assumption.

Where is the mistake in this proof?

Exercise 5 (Bridg-it on Graphs)

Consider the following game on a graph $G$. There are two players, a red color player $R$ and a blue color player $B$. Initially all edges of $G$ are uncolored. The two players alternately color an uncolored edge of $G$ with their color until all edges are colored. The goal of $B$ is that in the end, the blue-colored edges form a connected spanning subgraph of $G$. The goal of $R$ is to prevent $B$ from achieving his goal. Assume that $R$ starts the game.

Note that it was essentially shown in the lecture that $B$ can always win if $G$ contains two edge-disjoint spanning trees. (Recall the winning strategy for Player 1 in Bridg-it.)

Prove that on the other hand, $R$ can always win if $G$ does not contain two edge-disjoint spanning trees!

Discussion of the solution in the exercise class on 21.2.2013.