Graphs and Algorithms

Exercise 1 (Combining \(k\)-connected graphs)
Let \(G = (V, E)\) be the combined graph, as given in the exercise, and let \(S \subseteq V\) be an arbitrary subset of at most \(k - 1\) vertices (the set of removed vertices). Since \(G_1\), \(G_2\) and \(K_k\) are \(k\)-connected, then \(G_1[V \setminus S]\), \(G_2[V \setminus S]\) and \(K_k[V \setminus S]\) are connected. Thus it remains to show that for any two vertices \(a \in V(G_1) \setminus S\) and \(b \in V(G_2) \setminus S\) there exists a path which avoids \(S\) (we will implicitly prove that this also holds if either \(a\) or \(b\) are in \(V(K_k) \setminus S\)).

Since we removed at most \(k - 1\) vertices and there are \(k\) edges connecting \(G_1\) and \(K_k\) we know that there is at least one edge \(e = \{a', k'\}\) where \(a' \in V(G_1) \setminus S\) and \(k' \in V(K_k) \setminus S\). Now since \(G_1 \setminus S\) is still connected there exists a path \(P'\) from \(a\) to \(a'\). By a symmetric argument there exists an edge \(e' = \{k'', b''\}\), with \(k'' \in V(K_k) \setminus S\) and \(b'' \in V(G_2) \setminus S\), and a path \(P''\) which connects \(b\) to \(b''\). Since \(K_k \setminus S\) is connected we have that there is a path \(P'''\) connecting \(k'\) and \(k''\). Concatenating \(P'\), \(P'''\) and \(P''\) now yields a path from \(a\) to \(b\) proving that the graph is still connected.

Exercise 2 (Characterizations of forests)
(a) \(\Rightarrow\) (d): For ease of notation let’s denote the number of components of \(G\) by \(c(G)\). We proceed by induction on \(|E(G)|\). For the base case when \(|E(G)| = 0\) every vertex is isolated and we have \(|V|\) components which equals \(|V(G)| - |E(G)|\).

Now let’s assume that \(|E(G)| > 0\) and let \(e \in E(G)\) be an arbitrary edge. Since \(e\) is a bridge, \(G - e\) has one more component than \(G\) so the induction hypothesis now yields that

\[
c(G) = c(G - e) - 1 = |V(G - e)| - |E(G - e)| - 1 = |V(G)| - |E(G)|\]

which proves that the induction hypothesis holds.

(b) \(\Rightarrow\) (a): Let’s assume that every connected subgraph of \(G\) is induced. For the sake of contradiction let’s assume that \(G\) has a cycle with vertices from the set \(S = \{v_0, v_1, \ldots, v_t\}\) such that \(\{v_i, v_{i+1}\} \in E\) for \(i \in \{0, 1, \ldots, t-1\}\), and \(\{s_0, s_t\} \in E\). Now \(G[S]\) contains a cycle, but \(G[S] - e\), where \(e\) is an edge on the cycle, is connected but not induced which contradicts the assumption, thus \(G\) is a forest.
(c)⇒(b): Let’s assume that every induced subgraph has a leaf. Again for the sake of contradiction let’s assume there exists a connected subgraph $H$ of $G$ which is not induced. This in turn implies that $G[V(H)]$ contains a cycle because if $G$ were a forest then every induced subgraph on $G$ would also be a forest and hence every connected subgraph of $G$ would be induced (proving that (a) implies (b)). Now let $C$ be the vertices on a cycle in $G[V(H)]$, then $G[C]$ contains a cycle and each vertex in $G[C]$ has degree at least two contradicting the assumption.

(d)⇒(c): Let’s assume that the number of components is equal to $|V| - |E|$. For the sake of contradiction let’s assume that there exists an induced subgraph $G[S]$ which does not have a leaf. This in turn implies that there exists an edge $e \in G[S]$ which is not a bridge. Let’s consider $G \setminus e$. If it still contains a non-bridge edge we remove it and iterate until what remains is a graph where each edge is a bridge. Let $E'$ be the set of edges we removed, and note that $E' \neq \emptyset$. Then $G \setminus E'$ is a forest since each edge is a bridge and the number of components is $|V| - (|E| - |E'|) > |V| - |E|$, contradicting the assumption above.
Exercise 3 (No small cycles, few edges)

(a) By definition $G[V'']$ is connected. If $G[V'']$ contained a cycle the cycle would have length at most $2k + 1$ contradicting the fact that the girth of $G$ is at least $2(k + 1)$. To see this consider the spanning tree obtained by doing a BFS on $G[V'']$. If any edge $\{u, v\}$ is added to this tree then the unique paths to $u$ and $v$ in the spanning tree along with the edge will create a cycle of length at most $2k + 1$ contradicting the girth condition. Hence since $G[V'']$ is connected it is a tree.

(b) Now let’s consider the tree $G[V'']$ obtained in part (b). We will now give a lower bound on the number of vertices in the $G[V'']$. Note that since the minimum degree of $G[V'']$ is at least $\rho$ the root has at least $\rho$ children and each child has at least $\rho - 1$ children. Thus in the tree we have for $\rho > 2$ at least

\[ 1 + \rho + \rho (\rho - 1) + \cdots + \rho (\rho - 1)^{k-1} = 1 + \rho \left( \frac{(\rho - 1)^k - 1}{\rho - 2} \right) \]

vertices (the sum is geometric which yields the formula).

(c) Now if $\rho \leq 2$ the bound trivially holds so we can assume that $\rho > 2$. We now have using the result from part (b) that

\[
\begin{align*}
    n &\geq 1 + \frac{m}{n} \left( \frac{(\frac{m}{n} - 1)^k - 1}{\frac{m}{n} - 2} \right) \\
    \Rightarrow m - 2n &\geq \frac{m}{n} \left( \frac{m}{n} - 1 \right)^k - 2 \\
    \Rightarrow n - 2n \cdot \frac{n - 1}{m} &\geq \left( \frac{m}{n} - 1 \right)^k \\
    \Rightarrow n &\geq \left( \frac{m}{n} - 1 \right)^k \\
    \Rightarrow n^{\frac{1}{k}} + 1 &\geq \frac{m}{n}.
\end{align*}
\]

Which is the desired result.

Exercise 4 (Larger bipartite subgraph)

Let $G = (V, E)$ be a connected graph on $n$ vertices and $m$ edges. As suggested in the exercise, we will prove by induction on $n$ that $b(G) \geq \frac{m}{n} + \frac{n - 1}{4}$. For the base case $n = 1$ the claim trivially holds, and let us assume that it holds for all $n' < n$.

Proof of part (a). Let $v \in V$ be an articulation point of $G$, and let $C_1, \ldots, C_t \subseteq V$, where $t \geq 2$, be vertices of connected components of $G - v$. Since $|C_i| \geq 1$ for every $i \in [t]$, we also have $|C_i \cup v| \leq n - 1$. Thus, by induction hypothesis we have $b(G[C_i \cup v]) \geq \frac{m_i}{n_i} + \frac{n_i - 1}{4}$ for every $i \in [t]$, where $n_i$ and $m_i$ are the number of vertices and edges in $G[C_i \cup v]$, respectively. Simple calculation now yields that

\[
b(G) \geq \sum_{i=1}^{t} \frac{m_i}{2} + \frac{n_i - 1}{4} = \sum_{i=1}^{t} \left( \frac{m_i}{2} + \frac{n_i}{4} \right) - \frac{t}{4}.
\]
Further, observe that every edge of $G$ belongs to exactly one induced subgraph $G[C_i \cup v]$, thus $\sum_{i=1}^{t} m_i = m$. Finally,

$$\sum_{i=1}^{t} n_i = \sum_{i=1}^{t} (|C_i| + 1) = n - 1 + t,$$

hence we get

$$b(G) \geq \sum_{i=1}^{t} \left( \frac{m_i}{2} + \frac{n_i}{4} \right) + \frac{t}{4} \geq \frac{m}{2} + \frac{n - 1 + t - t}{4} = \frac{m}{2} + \frac{n - 1}{4}.$$  

\[\Box\]

Proof of part (b). Let $v \in V$ be a vertex of $G$ such that $v$ has an odd degree and is not articulation point. Since $G - v$ is connected, by induction we have $b(G - v) \geq \frac{m - \deg(v)}{2} + \frac{n - 2}{4}$, and further let $A, B \subseteq V \setminus \{v\}$, $A \cap B = \emptyset$, be partitions which prove this. Since $\deg(v)$ is odd, we either have $|\Gamma(v) \cap A| \geq \frac{\deg(v) + 1}{2}$ or $|\Gamma(v) \cap B| \geq \frac{\deg(v) + 1}{2}$, thus we can either add $v$ to $A$ or $B$ such that the number of additional edges in the bipartite subgraph is at least $\frac{\deg(v) + 1}{2}$. Together with the induction hypothesis this yields

$$b(G) \geq b(G - v) + \frac{\deg(v) + 1}{2} \geq \frac{m - \deg(v)}{2} + \frac{n - 2}{4} + \frac{\deg(v) + 1}{2} = \frac{m}{2} + \frac{n - 1}{4},$$

which proves the part (b) (note that we get a slightly stronger result which we need for part (c)).  

\[\Box\]

Proof of part (c). If $G$ has a vertex which either satisfies requirement of (a) or (b), we are done. Therefore, we can assume that all vertices in $G$ have an even degree, and further contains no articulation point. Our idea is to find two vertices $u, v \in V$ such that $\{u, v\} \in E$ and further $G[V \setminus \{u, v\}]$ is a connected graph. Let us for the moment assume that we have such two vertices. Then $G[V \setminus \{u\}]$ is a connected graph which contains a vertex $v$ such that $v$ has an odd degree and is not an articulation point. Thus from the part (b) and $|V \setminus \{u\}| = n - 1$ we have

$$b(G - u) \geq \frac{m - \deg(u)}{2} + \frac{n - 1}{4}.$$  

Similarly as in the previous part, we can add vertex $u$ such that at it "contributes" at least $\frac{\deg(u)}{2}$ additional edges in the bipartite subgraph, and therefore

$$b(G) \geq b(G - u) + \frac{\deg(u)}{2} \geq \frac{m - \deg(u)}{2} + \frac{n - 1}{4} + \frac{\deg(u)}{2} = \frac{m}{2} + \frac{n - 1}{4}.$$  

Thus it only remains to show that we can always find such vertices $u$ and $v$. Let $u \in V$ be any vertex and consider the block graph of $G - u$. Observe that the trivial case when the whole graph $G - u$ is one block easily implies that $u$ together with any neighbor satisfies the requirement. Thus we can assume that $G - u$ has at least one articulation point, and let $b$ be a block and $a$ an
articulation point such that $b$ is a leaf and $\{a,b\}$ is an edge in the block graph. Further, let $B$ be vertices associated with the block $b$, without the vertex $a$, and note that $|B| \geq 1$. Since $a$ is an articulation point in $G - u$, and $G$ is the connected graph without an articulation point, there has to exists an edge $\{v,u\} \in E$ such that $v \in B$. Since $v$ is not an articulation point in $G - u$, we conclude that $G[V - \{u,v\}]$ remains connected, thus proving that $v,u$ satisfy requirements.

Remark: $b(G) \geq \frac{m}{2} + \frac{n-1}{4}$ is known as the Edwards-Erdős bound.