Graphs and Algorithms

Exercise 1 (Two-way Hall’s condition)

Let $G = (A \cup B, E)$ be a bipartite graph with partitions $A$ and $B$ and $|A| = |B| = n$. Show that for every integer $k \in [0, n]$ the following conditions are sufficient for the existence of a perfect matching in $G$:

1. for all $S \subseteq A$ of size $|S| \leq k$ we have $|N(S)| \geq |S|$,
2. for all $S \subseteq B$ of size $|S| \leq n - k$ we have $|N(S)| \geq |S|$.

Note that the case $k = n$ is equivalent to the Hall’s theorem discussed in the lecture.

Exercise 2 (Star Matching)

Let $G = (A \cup B, E)$ be a bipartite graph with partitions $A$ and $B$ and $n = |A| \geq |B|$. Furthermore, let $v_1, \ldots, v_n$ be an arbitrary enumeration of the vertices from $A$ and $d_1, \ldots, d_n$ be integers. We say that a subgraph $M \subseteq G$ is a $(d_1, \ldots, d_n)$-matching if $\deg_M(v_i) = d_i$ for every $v_i \in A$ and $\deg_M(u) = 1$ for every $u \in B$.

Show that the following condition is sufficient for the existence of a $(d_1, \ldots, d_n)$-matching:

for all $S \subseteq A$ we have $|N(S)| \geq \sum_{v_i \in S} d_i$.

Remark: Try drawing one such matching to see why it’s called "star" matching.

Exercise 3 ($k$-orientation)

Let $k \in \mathbb{N}$, and consider a graph $G = (V, E)$ which satisfies the property that the number of edges in $G[V']$ is at most $k \cdot |V'|$, for every $V' \subseteq V$. Prove that the edges of $G$ can be oriented such that the outdegree of every vertex is at most $k$.

Hint: One way to solve this problem is to construct a certain bipartite graph and use Hall’s theorem to show that it has a saturating matching. Then deduce the orientation of edges from this matching.

Discussion of the solution in the exercise class on 17.4.2014.