As previously calculated, which consists of exactly $x, y$ and $z$, following bijection: map a forest $T \in T$ to a tree $n, R$ if $y$. Note, as previously defined as $n, R$. Let $x, u$ and $v$ are two vertices from $S_u \cup S_v$ or $u, v$. In particular, we have $d_G(y, u) \leq 1$ if $y \in S_u$ or $d_G(y, u) \leq 2$, if $y \in S_v$. Therefore, $d_G(x, y) \leq d_G(x, u) + d_G(y, u) \leq 3$. Same will hold for any $x \in S_v$ and $y \in V$ by symmetry. This concludes the proof.

**Exercise 2 (Number of labeled forests)**

Let $F(n, R)$, where $R \subseteq [n]$, be the family of all forests on the vertex set $[n]$ which consists of exactly $|R|$ trees. Similarly, let $T(n, R)$ be the family of all trees on the vertex set $[n + 1]$ with the property that $N_T(n + 1) = R$ for every $T \in T(n, R)$, i.e. the neighborhood of the vertex labeled with $n + 1$ is exactly the set of vertices in $R$. It is easy to see that $|F(n, R)| = |T(n, R)|$ by the following bijection: map a forest $F = (V, E) \in F(n, R)$ to a tree $T \in T(n, R)$ defined as

$$T = (V \cup \{n + 1\}, E \cup \bigcup_{i \in R} \{n + 1, i\}).$$

Therefore, instead of calculating $F(n, k) = |F(n, \{n - k + 1, \ldots, n\})|$ we will calculate $|T(n, \{n - k + 1, \ldots, n\})|$. First, observe that by the symmetry we have $|T(n, R)| = |T(n, R')|$ for every $R, R' \subseteq [n]$ with $|R| = |R'|$. In particular, we have $F(n, k) = |T(n, R)|$ for every $R \subseteq [n]$, such that $|R| = k$. Next, let us denote with $T_k(n)$ the family of all trees on the vertex set $[n + 1]$ such that $\deg_T(n + 1) = k$ for every $T \in T_k(n)$. Then

$$T_k(n) = \bigcup_{R \subseteq [n], |R| = k} T(n, R)$$
and thus by the previous observation we have
\[ F(n, k) = \frac{|T_k(n)|}{\binom{n}{k}}. \] (1)

It remains to calculate $|T_k(n)|$. This, however, easily follows from the fact that the degree of the vertex $n+1$ is $k$ in $T$ if and only if the number $n+1$ appears exactly $k-1$ times in the Prüfer code of $T$. Therefore, $|T_k(n)|$ is equal to the number of words from $[n+1]^{n-1}$ with the property that $n+1$ appears exactly $k-1$ times. We can choose such $k-1$ occurrences in $\binom{n-1}{k-1}$ ways and then choose the remaining elements in $n^{n-k}$ ways. Plugging this back in (1), we have
\[
F(n, k) = \frac{(n-1)^{n-k}}{\binom{n}{k}} = \frac{n}{k} \cdot \frac{k}{n} \cdot \frac{n^{n-k}}{\binom{n}{k}} = kn^{n-k-1},
\]
where we used the fact that $\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$. This finishes the proof.

**Exercise 3 (Strongly connected subgraph)**

First, let us assume the following claim.

**Claim 1.** Let $G = (V, E)$ be a strongly-connected digraph and $r \in V$ an arbitrary vertex from $V$. Then there exists a set $E' \subseteq E$ with $|E'| \leq |V| - 1$ such that for every $x \in V$ there is a directed path from $x$ to $r$ using only arcs in $E'$.

Let $G = (V, E)$ be a strongly-connected digraph and $r \in V$ an arbitrary vertex. We define a digraph $G^T = (V, E^T)$ as
\[ E^T = \{(x, y) \mid x, y \in V \text{ and } (y, x) \in E\}. \]

Note that the digraph $G^T$ is also a strongly-connected as any directed path from $x$ to $y$ in $G^T$ corresponds to a directed path from $y$ to $x$ in $G$.

Let $E_1 \subseteq E$ be a subset obtained by applying Claim 1 to the digraph $G$ and the vertex $r$. Thus, we know that there exists a directed path from any vertex $x \in V$ to $r$ using only arcs in $E_1$. Similarly, let $E_2^T \subseteq E^T$ be a subset obtained by applying Claim 1 to the digraph $G^T$ and the vertex $r$ and define $E_2 \subseteq E$ as
\[ E_2 = \{(x, y) \mid x, y \in V \text{ and } (y, x) \in E_1 \}. \]

Observe that there exists a directed path from each $x \in V$ to $r$ using only arcs in $E_2^T$, thus by the definition of $G^T$ it follows that there exists a directed path using only arcs in $E_2$ from $r$ to $x$ as well. Therefore, the digraph $(V, E_1 \cup E_2)$ is strongly-connected and since $|E_1 \cup E_2| \leq 2(|V| - 1)$, this concludes the proof.

What is left is to prove Claim 1.

**Proof of Claim 1.** We will iteratively define sets $W \subseteq V$ and $H \subseteq E$. The set $E'$ will be the set $H$ at the end of the process.

In the beginning let $W = \{r\}$ and $H = \emptyset$ and $U := V \setminus W$. As long as there $U \neq \emptyset$ we repeat the following. Take an arbitrary $x \in U$ and let $P = (x, a_1, \ldots, a_{k-1}, a_k)$, where $a_k = r$, be any directed path in $G$ from $x$ to $r$. Let
\( i \in [k] \) be the smallest index such that \( a_i \in W \). There must exist such a vertex as \( r \in W \). We update \( W \) and \( H \) in the following way:

\[
W = W \cup \{x, a_1, \ldots, a_{i-1}\} \quad \text{and} \quad H = H \cup \{(r, a_1), (a_1, a_2), \ldots, (a_{i-1}, a_i)\}.
\]

We prove by induction on the size of \( W \) that \(|H| = |W| - 1\) and that all vertices in \( W \) have a directed path to \( r \) using only arcs in \( H \). Base of the induction holds trivially as \( W = \{r\} \).

Assume that \( W \) has the desired properties and \( U \neq \emptyset \). Let us denote by \( \forall \) and \( \exists \) sets \( W \) and \( H \) after the next update. When \( W \) and \( H \) are updated, we always add the same number of vertices and arcs and hence \(|H'| = |W'| - 1\) holds. Let us assume that \( W' = W \cup \{x, a_1, \ldots, a_{i-1}\} \) and \( H' = H \cup \{(r, a_1), (a_1, a_2), \ldots, (a_{i-1}, a_i)\} \). By the induction hypothesis, vertex \( a_i \) has a directed path to \( a_i \) using arcs in \( H \). As all the newly inserted vertices \( \{x, a_1, \ldots, a_{i-1}\} \) have a path to \( a_i \) using arcs \( \{(r, a_1), (a_1, a_2), \ldots, (a_{i-1}, a_i)\} \), there is also a path to \( r \) using arcs in \( H' \), which concludes the proof.

Finally, as we stop when \( U \) is empty, clearly \( W \) will be equal to \( V \) and \( E' := H \) will have the correct size.

\[\square\]

**Exercise 4 (Minimum degree - maximum forward degree)**

Let \( G = (V, E) \) be a given graph with \( V = \{v_1, \ldots, v_n\} \). We first show that

\[
\max_{U \subseteq V} \delta(G[U]) \geq \min_{\pi \in S_n} \max_{v \in V} fdeg_{\pi}(v). \tag{2}
\]

To show \(\ref{eq:2}\), note it suffices to construct a permutation \( \pi \in S_n \) which satisfies

\[
\max_{U \subseteq V} \delta(G[U]) \geq \max_{v \in V} fdeg_{\pi}(v). \tag{3}
\]

This can be done as follows. Set \( W := V \) and as long as \( W \neq \emptyset \) remove an element \( w \in W \) from \( W \) which satisfies \( \deg_{\pi}[W](w) = \delta(G[W]) \). Observe that it is not possible that some vertex \( w \in V \) was removed twice and moreover, since \( W = \emptyset \) at the end, we have that each vertex from \( V \) was removed exactly once. Following the order in which vertices were removed gives us as an ordering \( (v_1, \ldots, v_n) \) of the vertices in \( V \). We claim that \( \pi = (i_1, \ldots, i_n) \) satisfies \(\ref{eq:3}\).

It follows from the construction that for each \( j \in [n] \) we have

\[
\delta(G_j) = \deg_{G_j}(v_{i_j}) = fdeg_{\pi}(v_{i_j}),
\]

where \( G_j := G[\{v_{i_1}, \ldots, v_{i_n}\}] \). In particular, this implies

\[
\max_{U \subseteq V} \delta(G[U]) \geq \max_{j \in [n]} \delta(G_j) = \max_{j \in [n]} fdeg(v_{i_j}),
\]

which proves \(\ref{eq:3}\).

Next, we prove

\[
\max_{U \subseteq V} \delta(G[U]) \leq \min_{\pi \in S_n} \max_{v \in V} fdeg_{\pi}(v). \tag{4}
\]
Let $\pi = (i_1, \ldots, i_n) \in S_n$ be an arbitrary permutation and consider a subset $W \subseteq V$ such that $\delta(G[W]) = \max_{U \subseteq V} \delta(G[U])$ (graphs we consider are finite, thus such $W$ exists). Let $j \in [n]$ be the smallest index such that $v_{i_j} \in W$. Since all neighbors of $v_{i_j}$ come after the vertex $v_{i_j}$ in the permutation $\pi$, we have

$$d_{G}(v_{i_j}) = d_{G,\pi}(v_{i_j}) = f\deg(\pi)(v_{i_j}),$$

where $G$ is as defined before. Finally, by the choice of $W$ and $v_{i_j}$ we have

$$\max_{v \in V} f\deg(\pi)(v) \geq d_{G}(v_{i_j}) = d_{G,\pi}(v_{i_j}) \geq \delta(G[W]) = \max_{U \subseteq V} \delta(G[U]).$$

Since this holds for an arbitrary permutation $\pi \in S_n$, the inequality\(^1\) follows. Together with\(^2\), this implies

$$\max_{U \subseteq V} \delta(G[U]) = \min_{\pi \in S_n} \max_{v \in V} f\deg(\pi)(v).$$

**Exercise 5 (Diameter and many $k$-paths)**

Let $T$ be a tree on $n$ vertices and $k \in \mathbb{N}$ such that $q := \text{diam}(T) \geq 2k - 3$. We show that $T$ contains at least $n - k$ different paths of length $k$.

In order to make the proof easier to follow, we first introduce some notation. For two distinct vertices $v_1, v_2 \in V(T)$, we denote with $P(v_1, v_2)$ the unique path from $v_1$ to $v_2$ in $T$. Furthermore, for a given set of vertices $S \subseteq V(T)$ we define $d_S(v)$ to be the minimum distance between a vertex $v \in V(T)$ and some vertex in $S$, i.e. $d_S(v) := \min_{w \in S} d(v, w)$.

For $k = 1$, the statement follows from the fact that $T$ contains exactly $n - 1$ edges and thus $n - 1$ different paths of length $1$. Therefore, from now on we can assume that $k \geq 2$.

Let $v_1, v_{q+1} \in V(T)$ be a pair of vertices with $d(v_1, v_{q+1}) = q$, let $P(v_1, v_{q+1}) = \{v_1, v_2, \ldots, v_{q+1}\}$ and $S := \{v_1, \ldots, v_{q+1}\}$ be the set of vertices of the path $P(v_1, v_{q+1})$. Observe that there are exactly $q - k + 1$ different subpaths of the path $P(v_1, v_{q+1})$ of length $k$:

$$P(v_1, v_{k+1}), P(v_2, v_{k+2}), \ldots, P(v_{q-k}, \ldots, v_q), P(v_{q-k+1}, v_{q+1}).$$ (5)

Next, for each vertex $v \in V(T) \setminus S =: R$ we define a path $p_v$ as follows: if $d(v, p_1) \geq k$ then set $w := v_1$ and otherwise $w := v_{q+1}$ and let $p_v$ be a subpath of the path $P(v, w)$ which contains the vertex $v$ and is of length exactly $k$. We claim that such paths are well defined and that they are all different (and also different from paths defined in (5)). Since there are $n - q - 1$ such paths, together with paths defined in (5) this gives $n - q - 1 + q - k + 1 = n - k$ different paths of length $k$.

First, observe that $d(v, p_1) \geq k$ or $d(v, p_{q+1}) \geq k$. Assuming otherwise, there exists a walk from $p_1$ to $p_{q+1}$ which goes through the vertex $v$ and is of length at most $d(v, p_1) + d(v, p_{q+1}) \leq 2k - 2$. However, since $v \notin S$, the vertex $v$ is not part of the path $P(v_1, v_{q+1})$, the unique path from $v_1$ to $v_{q+1}$. Therefore, such a walk contains at least two edges which are not contained in $P(v_1, v_{q+1})$ and transforming it into a path implies that $|P(v_1, v_{q+1})| \leq 2k - 2 < 2k - 3$. 


which is a contradiction with the choice of \( v_1 \) and \( v_{q+1} \). Therefore, if \( d(v, p_1) < k \) then \( d(v, p_{q+1}) \geq k \) and \( P(v, w) \) is always of length at least \( k \). Then there also exists a subpath of the path \( P(v, w) \) which contains the vertex \( v \) and is of length exactly \( k \), thus \( p_v \) is well-defined.

Next, we show that all such obtained paths are different. First, observe that for each \( v \in R \) the path \( p_v \) contains the vertex \( v \). Therefore, it also contains an edge which is not part of \( P(v_1, v_{q+1}) \) and thus is different from all the paths defined in \( \textbf{[5]} \). It remains to show that \( p_v \neq p_w \) for all distinct \( v, w \in R \). This easily follows from the following claim: for every \( v \in R \) we have that the endpoint \( w \) of the path \( p_v \), different from \( v \), satisfies

\[
d_S(w) < d_S(v).
\]

Thus if \( p_v = p_w \) for some \( v \neq w \) then by the previous claim \( d_S(w) < d_S(v) \) and \( d_S(w) > d_S(v) \), which is clearly not possible. Therefore, in order to finish the proof of the exercise it suffices to prove the claim.

Let us consider some vertex \( v \in R \) such that \( d_S(v) = i \), for some \( i > 0 \). We first prove that \( P(v, v_1) = (v = p_1, \ldots, p_k = v_1) \) is such that \( d_S(p_j) \geq d_S(p_{j+1}) \) for every \( j \in [k-1] \), with equality if and only if \( d_S(p_{j+1}) = 0 \). Note that there exists a unique vertex \( v_t \in S \) such that \( d(v, v_t) = i \) and consider the path \( P(v, v_t) = (v = u_1, u_2, \ldots, u_k = v_t) \). It is easy to see that along this path we have \( d_S(u_j) < d_S(u_{j+1}) \) for every \( j \in [k-1] \). Therefore, the path \( (v = u_1, \ldots, u_k = v_t, v_{t-1}, \ldots, v_1) \) is a path from \( v \) to \( v_1 \) and it satisfies the desired property. In the similar way we deduce that \( P(v, v_{q+1}) \) satisfies the same property. Therefore, since \( p_v \) is a subpath of either \( P(v, v_1) \) or \( P(v, v_{q+1}) \) and \( d_S(v) > 0 \), we have that the endpoint \( w \) of \( p_v \), different from \( v \), satisfies \( d_S(w) < d_S(v) \). This finishes the proof of the claim.