Graphs and Algorithms

Exercise 1 (Real matrices)

Let $M$ be a matrix as stated in the exercise and consider the bipartite graph $G = (V,E)$ with

$$V := \{r_i\}_{i=1}^n \cup \{c_i\}_{i=1}^n \quad \text{and} \quad E := \{(r_i, c_j) \mid m_{i,j} \geq 1/n^2\}.$$  

The main observation is that if $G$ contains a perfect matching then there exists a desired permutation. To see that, let us assume that $P = \{e_1, \ldots, e_n\}$ is a subset of the edges of $G$ which form a perfect matching, with $e_i = \{r_i, c_j\}$. Then setting $\pi(i) := j_i$ satisfies the desired properties: since $P$ is a matching we have that $\pi$ is a bijection and $e_i \in E(G)$ implies $m_{i,j_i} \geq 1/n^2$.

We show that $G$ contains a saturating matching for $\{r_i\}_{i=1}^n$ using Hall’s theorem. As partitions are of the same size, this implies the existence of a desired perfect matching. Let $S \subseteq \{r_i\}_{i=1}^n$ be an arbitrary subset. Then

$$\sum_{r_i \in S} \sum_{c_j \in \Gamma(S)} m_{i,j} = \sum_{r_i \in S} \left( \sum_{j=1}^n m_{i,j} - \sum_{c_j \in \Gamma(S)} m_{i,j} \right) > \sum_{r_i \in S} (1 - n/n^2) = |S|(1 - 1/n) \geq |S| - 1. \quad (1)$$

In the first inequality we used the fact that the elements of every row sum up to 1 and $c_j \notin \Gamma(S)$ (thus $c_j \notin \Gamma(r_i)$) implies that $m_{i,j} < 1/n^2$. The second inequality follows from trivial observation that $|S| \leq n$. On the other hand we have

$$\sum_{r_i \in S} \sum_{c_j \in \Gamma(S)} m_{i,j} = \sum_{c_j \in \Gamma(S)} \sum_{r_i \in S} m_{i,j} \leq \sum_{c_j \in \Gamma(S)} \sum_{i=1}^n m_{i,j} = |\Gamma(S)|, \quad (2)$$

where we simply used the fact that the elements of every column sum up to 1. Finally, from $(1)$ and $(2)$ we have

$$|S| - 1 < \sum_{r_i \in S} \sum_{c_j \in \Gamma(S)} m_{i,j} \leq |\Gamma(S)|,$$

or equivalently $|S| \leq |\Gamma(S)|$. Therefore, we have proven that $G$ satisfies Hall’s condition which, together with previous observations, finishes the proof of the exercise.
We define \( f \) as
\[
\text{The proof relies on the following claim.}
\]
we can assume that
\[
\text{Therefore we know }
\]
\( a \) connects
\[
\text{Part (i).}
\]
Let \( S \subseteq V \) be an arbitrary subset of vertices with \( |S| = 2k - 2 \). We will prove that \( S \) is not a separating set in \( G \) and as \( S \) is chosen arbitrarily this means that \( G \) is \((2k - 2)\)-connected.
Suppose that \( G[V \setminus S] \) is disconnected and let \( x \) and \( y \) be two vertices from different components of \( G[V \setminus S] \). Let \( (a_1, b_1), \ldots, (a_{k-1}, b_{k-1}) \) be arbitrary pairing of vertices in \( S \). As \( G \) is \( k \)-linked we know that for a set of pairs \( \{(x, y), (a_1, b_1), \ldots, (a_{k-1}, b_{k-1})\} \) we can find \( k \) vertex disjoint paths connecting each pair. In particular there is a path between \( x \) and \( y \) which avoids set \( S \). However, this is not possible as \( x \) and \( y \) are in separate components in \( G[V \setminus S] \).
\[
\text{Part (ii).}
\]
Let \( X = \{(a_1, b_1), \ldots, (a_k, b_k)\} \) an arbitrary set with all elements being vertices from \( G \) and \( a_i \neq b_i \) for all \( i \in [k] \), i.e. \( X \in \binom{V}{2}^k \). We say that \( X \) is pseudo \( k \)-linked if there exist \( k \) internally vertex disjoint paths \( P_1, \ldots, P_k \), where \( P_i \) connects \( a_i \) to \( b_i \) and avoids all vertices \( a_j \) and \( b_j \) which are different from \( a_i \) and \( b_i \).

The proof relies on the following claim.
\[
\text{Claim 1. Let } X = \{(a_1, b_1), \ldots, (a_k, b_k)\} \text{ be an arbitrary element of } \binom{V}{2}^k. \text{ Let us define a function } f: \binom{V}{2}^k \to \mathbb{N} \text{ in the following way}
\]
\[
f(X) := \left| \bigcup_{i=1}^{k} (a_i \cup b_i) \right|
\]
and suppose \( f(X) \leq 2k - 1 \). If for all \( Y \in \binom{V}{2}^k \) with \( f(Y) > f(X) \) it holds that \( Y \) is pseudo \( k \)-linked, then \( X \) is pseudo \( k \)-linked.

**Proof of the claim.** Assume that for all \( Y \in \binom{V}{2}^k \) with \( f(Y) > f(X) \) it holds that \( Y \) is pseudo \( k \)-linked.
As \( f(X) < 2k \) there must exist different \( i \) and \( j \) such that \( \{a_i, b_i\} \cap \{a_j, b_j\} \neq \emptyset \) and let \( v \) be an arbitrary element of \( \{a_i, b_i\} \cap \{a_j, b_j\} \). Without loss of generality we can assume that \( v = a_i \). As the graph \( G \) is \((2k - 1)\)-connected we know that \( \delta(G) \geq 2k - 1 \). In particular \( \deg(v) \geq 2k - 1 \) and therefore there exists a vertex \( w \in N(v) \setminus (\bigcup_{i=1}^{k} (a_i \cup b_i)) \).

We define \( X' \) in the following way
\[
X' = (X \setminus (a_i, b_i)) \cup (w, b_i).
\]
By the construction of \( X' \) and our choice of \( w \) we know that \( f(X') > f(X) \).
Therefore we know \( X' \) is pseudo \( k \)-linked. In other words, there exist \( k \) vertex disjoint paths \( P_1, \ldots, P_k \) where \( P_i \) connects \( a_i \) to \( b_i \) and avoids all vertices \( a_j \) and \( b_j \) which are different from \( a_i \) and \( b_i \). As \( v \) is already a member of some pair from \( X' \) we know that it doesn’t appear as an internal vertex in any of the paths \( P_1, \ldots, P_k \). Moreover, as \( w \) is a unique vertex in \( X' \) it doesn’t appear in any
As for sets $X \in \binom{V}{2}^k$ with $f(X) = 2k$ we directly have that $X$ is pseudo $k$-linked, by using the Claim 1 we have that all $X \in \binom{V}{2}^k$ are pseudo $k$-linked, which concludes the proof.

**Exercise 3 (Strong Hall’s condition)**

Throughout the proof, whenever we refer to a graph $G$ we assume that it is a bipartite graph with partitions $A$ and $B$. We call a subset of vertices $S \subseteq A$ critical if $|\Gamma(S)| = \sigma(G) + |S|$.

**Part (i).**

Let $X, Y \subseteq A$ be two critical subsets with a non-empty intersection, i.e. $X \cap Y \neq \emptyset$. We show that the union and intersection of two such critical sets are also critical.

First, from the inclusion-exclusion principle we have

$$|X \cap Y| + |X \cup Y| = |X| + |Y|.$$  

(3)

Further, note that if $b \in \Gamma(X \cap Y)$ then $b \in \Gamma(a)$ for some $a \in X \cap Y$, thus $b \in \Gamma(X) \cap \Gamma(Y)$ and so $\Gamma(X \cap Y) \subseteq \Gamma(X) \cap \Gamma(Y)$. Together with a trivial observation that $\Gamma(X \cup Y) = \Gamma(X) \cup \Gamma(Y)$, we get

$$|\Gamma(X \cup Y)| + |\Gamma(X \cap Y)| \leq |\Gamma(X) \cup \Gamma(Y)| + |\Gamma(X) \cap \Gamma(Y)|$$

$$= |\Gamma(X)| + |\Gamma(Y)|,$$  

(4)

where the equality again follows from the inclusion-exclusion principle. Finally, subtracting (3) from (4) we get

$$|\Gamma(X) \cup \Gamma(Y)| - |X \cup Y| + |\Gamma(X) \cap \Gamma(Y)| - |X \cap Y|$$

$$\leq |\Gamma(X)| - |X| + |\Gamma(Y)| - |Y| = 2\sigma(G).$$

Since $|\Gamma(X \cup Y)| - |X \cup Y| \geq \sigma(G)$ and $|\Gamma(X \cap Y)| - |X \cap Y| \geq \sigma(G)$ (from the definition of $\sigma(G)$ and the fact that $X \cap Y \neq \emptyset$) it follows that both $X \cup Y$ and $X \cap Y$ are critical.

**Part (ii)**

Let $G$ be as stated in the exercise. Consider an arbitrary vertex $v \in A$. Our goal is to show that $\deg(v) = \sigma(G) + 1$. Observe that it trivially follows from the definition of $\sigma(G)$ that $\deg(v) \geq \sigma(G) + 1$. Thus it remains to show the upper bound.

It follows from the edge-minimality of $G$ (property described in the exercise) that for every vertex $u \in \Gamma(v)$ we have $\sigma(G_u) < \sigma(G)$, where $G_u$ is obtained from $G$ by deleting the edge $\{v, u\}$. Let us consider an arbitrary vertex $u \in \Gamma(v)$ and let

$$3$$

$X_u \subseteq A$ be a critical set in $G_u$. Then $v \in X_u$ as otherwise $\Gamma_G(X_u) = \Gamma_{G_u}(X_u)$ since no edge touching $X_u$ was removed, thus

$$|\Gamma_{G_u}(X_u)| = |\Gamma_G(X_u)| = |X_u| + \sigma(G) > |X_u| + \sigma(G_u)$$

contradicting the choice of $X_u$. Further, note that $|\Gamma_G(X_u)| \leq |\Gamma_{G_u}(X_u)| + 1$ since we removed exactly one edge touching vertices in $X_u$, namely vertex $v$.

On the other hand, since $X_u$ is critical in $G_u$ we have

$$|\Gamma_{G_u}(X_u)| = \sigma(G_u) + |X_u| < \sigma(G) + |X_u| \leq |\Gamma_G(X_u)|.$$

Therefore, we can conclude that $|\Gamma_G(X_u)| = |\Gamma_{G_u}(X_u)| + 1$ and thus $u \notin \Gamma(X_u \setminus \{ v \})$. Since $\sigma(G_u)$ and $\sigma(G)$ are integers we can have that $X_u$ is critical in $G$.

To summarize, for each $u \in \Gamma(v)$ there exists a set $X_u \subseteq A$ such that $X_u$ is critical in $G$, contains the vertex $v$ and $u \notin \Gamma(X_u)$. It follows now from the part (i) of the exercise that

$$S := \bigcap_{u \in \Gamma(v)} X_u$$

is a critical set in $G$ (as it is non-empty intersection of critical sets). Therefore

$$|\Gamma(S)| = \sigma(G) + |S| \text{ and } \Gamma(S \setminus \{ v \}) \cap \Gamma(v) = \emptyset. \text{ If } S = \{ v \} \text{ then we are done as we have } \deg(v) = \sigma(G) + 1. \text{ Otherwise, we have } S \setminus \{ v \} \neq \emptyset \text{ and }$$

$$|\Gamma(S \setminus \{ v \})| = |\Gamma(S)| - |\Gamma(v)| = \sigma(G) + |S| - \deg(v).$$

As $|\Gamma(S \setminus \{ v \})| \geq \sigma(G) + |S| - 1$, from the above equality we get

$$\sigma(G) + |S| - 1 \leq \sigma(G) + |S| - \deg(v)$$

and so $\deg(v) \leq 1 \leq \sigma(G) + 1$. Since $v$ is an arbitrary vertex, this finishes the proof.

**Part (iii)**

We first prove that if $\sigma(G) \geq 1$ then there exists a spanning forest $F \subseteq G$ such that $\deg_F(v) = 2$ for every $v \in A$. Let $F \subseteq G$ be a spanning subgraph (i.e. on the same set of vertices as $G$) such that $\sigma(F) = 1$ and $F$ is edge-minimal. In other words, $F$ is such that for every graph $G'$ obtained from $F$ by removing a single edge we have $\sigma(F) > \sigma(G')$. Note that such $F$ can be obtained from $G$ by removing edges until it satisfies the property.

Using part (iii) of the exercise, we have $\deg_F(v) = 2$ for every $v \in A$. Therefore, it remains to prove that $F$ is a forest. Assume towards the contradiction that $F$ contains a cycle $C = (v_1, \ldots, v_t, v_1)$. Since $F$ is a subgraph of the bipartite graph, $C$ is of even length and since the vertices have to alternate between sets $A$ and $B$ we have $|C \cap A| = |C \cap B|$. Furthermore, this also implies that every vertex $v \in C \cap A$ has degree 2 inside $C \cap B$. Thus $\Gamma(C \cap A) = C \cap B$ and so $|\Gamma(C \cap A)| = |C \cap A|$, contradicting the fact that $\sigma(G) \geq 1$. Therefore, $F$ is acyclic and thus a forest.

**Exercise 4 (Hypergraph coloring)**

Before we prove the lemma, let us see how it implies the existence of a desired coloring. Let $H = (S, E)$ be a hypergraph such that for every (non-empty)
subset of hyperedges \( E' \subseteq E \) we have
\[
|E'| < |\cup_{e \in E'} e|.
\] (⋆)

Then, according to the lemma, there exists a forest \( G = (S, E_g) \) with the property that for every hyperedge \( e \in E \) there exist two distinct members \( a_e, b_e \in e \) such that \( \{a_e, b_e\} \in E_g \). Since \( G \) is a forest, it contains no cycles – and thus no odd cycles, hence we know that \( G \) is also a bipartite graph. Let us denote partitions of \( G \) with \( R \) and \( B \) (note that the partitions might not be uniquely determined, in which case we consider an arbitrary one) and consider the coloring of \( H \) which assigns red to vertices in \( R \) and blue to vertices in \( B \). It is easy to see that such a coloring is rainbow: for every hyperedge \( e \in E \) it follows from \( \{a_e, b_e\} \in E_g \) that \( e \) contains two vertices which belong to different partitions of \( G \) and thus were assigned a different color.

Therefore, it remains to prove the lemma.

**Proof of the lemma.** Let \( H = (S, E) \) be a graph which satisfies property (⋆). Consider the incidence graph \( I = (E \cup S, E_I) \) of \( H \), i.e. a bipartite graph in which one partition corresponds to the set of hyperedges and the other to the set of vertices, with
\[
E_I := \{\{e, s\} \mid e \in E, s \in S, s \in e\}.
\]
From the definition of \( I \), for every \( E' \subseteq E \) we have
\[
\Gamma(E') = \{s \in S \mid \exists e \in E', s \in e\} = \cup_{e \in E'} e,
\]
thus it follows from (⋆) that \( |\Gamma(E')| \geq |E'| + 1 \). In the terminology of Exercise 3, we have that \( \sigma(I) \geq 1 \). Therefore, we can apply Exercise (3.iii) to obtain a spanning forest \( F \subseteq I \) such that \( \deg_F(e) = 2 \) for every \( e \in E \). We claim that the graph \( G = (S, E_g) \), with
\[
E_g := \{\Gamma_F(e) \mid e \in E\},
\]
satisfies the desired properties.

First, it follows from \( \deg_F(e) = 2 \) that \( G \) is indeed a graph, i.e. every edge has exactly two endpoints. Furthermore, since \( \Gamma_I(e) = e \) we have \( \Gamma_F(e) \subseteq e \) and thus for every hyperedge there exists two distinct vertices, namely those specified by \( \Gamma_F(e) \), which are forming an edge in \( G \). Finally, it remains to show that \( G \) is a forest. Assume towards the contradiction that there exists a cycle \( C = (v_1, \ldots, v_k, v_{k+1}) \) in \( G \), where \( v_1 = v_{k+1} \). Then for every pair of consecutive vertices \( (v_i, v_{i+1}) \), since they form an edge in \( G \) we know that there exists a hyperedge \( e_i \in E \) such that \( \{v_i, v_{i+1}\} = \Gamma_F(e_i) \). But then \( C' = (v_1, e_1, v_2, \ldots, v_k, e_k, v_{k+1}) \) is a cycle in \( F \), which is a contradiction with the fact that \( F \) is a forest. Therefore, we can conclude \( G \) is acyclic. This finishes the proof of the claim. \( \square \)