Graphs and Algorithms

Exercise 1 (Odd common neighbourhood)
As we know from the lecture, a graph $G = (V, E)$ is Eulerian if and only if every vertex $v \in V$ has an even degree. Thus, for a graph $G$ as given in the exercise, it suffices to prove that every vertex has an even degree.

Let $a \in V$ be an arbitrary vertex. If $V = \{a\}$ then we are already done. Otherwise, there exists a vertex $b \in V \setminus \{a\}$ and $|N_G(a) \cap N_G(b)|$ is an odd number. But then $N_G(a) \cap N_G(b) \neq \emptyset$ and so $N_G(a) \neq \emptyset$. Let us now consider the graph $G_a := G[N_G(a)]$. It is easy to see that for any vertex $b \in V(G_a)$ we have that $N_{G_a}(b) = N_G(a) \cap N_G(b)$ and therefore every vertex in $G_a$ has an odd degree (in $G_a$). As every graph has an even number of vertices with an odd degree, it follows that $|V(G_a)| = |N_G(a)|$ is an even number thus the vertex $a$ has an even degree in $G$.

Exercise 2 (Hamilton cycles and paths)
Let us first prove Lemma 1 from the exercise.

Proof of Lemma 1. Let $(v_1, \ldots, v_n)$ be an ordering of the vertices as given in the statement of the lemma. We first show that $G$ is connected. We prove by induction on $i$ that the vertex $v_i \in G$ belongs to a component with more than $n/2$ vertices. This implies that $G$ is connected similarly as in the proof of Dirac’s theorem. For $i \in \{n/2, \ldots, n\}$ it follows from the fact that $\deg(v_i) \geq n/2$ that the component of $v_i$ contains at least $n/2 + 1$ vertices. Assume further that the claim holds for all vertices $v_j$ with $j < i$, for some $i \leq n/2$, and consider the vertex $v_i$. By the assumption we have $\deg(v_i) \geq i + 1$, thus there exists an index $i' > i$ such that $\{v_{i'}, v_i\} \in E(G)$. But then, by the induction hypothesis, vertex $v_i$ belongs to a component with at least $n/2 + 1$ vertices, thus the same holds for $v_i$.

We now show that $G$ is Hamiltonian. Let us consider a longest path $P = (v_{i_1}, \ldots, v_{i_k})$ in $G$. We will show that there exists a cycle $C$ which covers all the vertices of $P$. Observe that if $P$ is not a Hamiltonian path, this gives a cycle of length $k < n$. Since the graph is connected, there exists an edge with one endpoint in $V(C)$ and the other in $V(G) \setminus V(C)$ thus giving a path larger than $P$. As this is a contradiction with the choice of $P$, the cycle $C$ has to be Hamiltonian. Now let us prove that such cycle always exists.

First, observe that $N(v_{i_1}), N(v_{i_k}) \subseteq V(P)$ as otherwise we could extend $P$, contradicting its choice. Further, without loss of generality we can assume that $i_1 < i_k$. If $i_1 \geq n/2$, then both endpoint have degree at least $n/2$ and the proof
We do a case distinction based on whether $i_1 < n/2$ and consider the set $A$ of the predecessors of the neighbors of $v_i$ on $P$, i.e.

$$A := \{i_1 \mid v_{i_1+1} \in N(v_i)\},$$

Note that $|A| \geq i_1 + 1$, thus there exists some $i_2 > i_1$ such that $i_2 \in A$. We can now "rotate" the path $P$ around $v_i$ to get a new path $P'$,

$$P' = (v_i, v_{i_2}, v_1, v_{i_2+1}, v_{i_3+2}, \ldots, v_{i_k}).$$

Observe that the other endpoint of $P'$ is the same as in $P$. Since the index of the "smaller" endpoint of the path increased by at least $1$ and the other endpoint remained the same, by repeating this procedure at most $n$ times we have that $\min\{i_1, i_k\} \geq n/2$. But then the proof again follows from the proof of Dirac’s theorem, thus there exists a desired cycle. 

Let us now turn to the part $(b)$ of the exercise. Let $G = (V, E)$ be a graph on $n$ vertices as given in the exercise and consider two arbitrary distinct vertices $a, b \in V$. We show that the graph $G'$ created from $G$ by adding a new vertex $x$ and connecting it to $a$ and $b$, i.e.

$$G' := (V \cup \{x\}, E \cup \{a, x\} \cup \{b, x\})$$

is Hamiltonian. Since the vertex $x$ has degree exactly 2 in $G'$, by removing it from a Hamiltonian cycle we get a Hamiltonian path which starts in one of the neighbors of $x$ and ends in the other. As the neighbors of $x$ are precisely vertices $a$ and $b$, this finishes the proof.

We do a case distinction based on whether $n$ is even or odd. Let us first assume that $n$ is even. Then every vertex $v \in V$ has degree at least $n/2 + 1$ in $G'$. Let $(u_1, \ldots, u_{n+1})$ be an ordering of vertices in $G'$ such that $u_1 = x$ and all other vertices are ordered arbitrarily. We claim that such an ordering satisfies the condition of Claim 1. Indeed, for $i = 1$ we have $d_{G'}(u_1) = 2$ as desired. For $i \in \{1, \ldots, n/2\}$ (note that $n/2 = \lceil(n+1)/2\rceil$) we have $d_{G'}(u_i) \geq n/2 + 1 \geq i + 1$. Similarly, for $i \in \{n/2 + 1, \ldots, n + 1\}$ we have $d_{G'}(u_i) \geq n/2 + 1 > (n + 1)/2$, as desired. Therefore, by Claim 1 the graph $G'$ is Hamiltonian.

Let us now assume that $n$ is odd. Then every vertex $v \in V$ has degree at least $(n + 1)/2$ and, moreover, vertices $a$ and $b$ have degree at least $(n + 1)/2 + 1$. Let now $(u_1, \ldots, u_{n+1})$ be an ordering of the vertices in $G'$ such that $u_1 = x$, $u_{(n+1)/2} = a$ and all other vertices are ordered arbitrarily. Similarly as in the previous case, the idea is to apply Claim 1 to deduce that $G'$ is Hamiltonian. Again, for $i = 1$ it holds that $d_{G'}(u_1) = 2$. For $i \in \{1, \ldots, (n + 1)/2 - 1\}$ we have $d_{G'}(u_i) \geq (n + 1)/2 > i + 1$. For $i = (n + 1)/2$ we have $u_i = a$ and thus $d_{G'}(u_i) \geq (n + 1)/2 + 1 \geq i + 1$. Finally, for $i \in \{(n + 1)/2, \ldots, n + 1\}$ we have $d_{G'}(u_i) \geq (n + 1)/2$ and therefore we can apply Claim 1. This finishes the proof.

**Exercise 3 (Planar graphs with large girth)**

If $G$ is a forest, one can easily check that the claim holds. Therefore let us assume that $G$ contains at least one cycle.
Let $V' \subseteq V$ be a set of vertices which are not part of any cycle and let

$$E' := \{ e \in E \mid e \text{ incident to some } v \in V' \}.$$ 

As each edge in $E'$ is incident to some $v \in V'$ the set of edges $E'$ cannot form a cycle. Therefore $E'$ forms a forest and hence $|E'| \leq |V'| - 1$. Note that set of edges in $G[V \setminus V']$ is exactly $E \setminus E'$.

We know that every edge of $G[V \setminus V']$ is a part of a cycle and moreover it is a part of exactly two cycles. On the other hand, each cycle contains at least five edges. Therefore, if by $F$ we denote the number of cycles in $G[V \setminus V']$ by double counting we have

$$5F \leq 2|E \setminus E'|.$$

By using the fact that $|V \setminus V'| - |E \setminus E'| + F = 2$ from Euler’s formula and the previous inequality we get

$$10 - 5|V \setminus V'| + 5|E \setminus E'| \leq 2|E \setminus E'|,$$

which after simple algebraical manipulations gives us

$$|E \setminus E'| \leq \frac{5|V \setminus V'|}{3} - \frac{10}{3}.$$ 

Next, add $\frac{5|V'|}{3}$ to both sides of the last equation to obtain

$$\frac{5|V \setminus V'| + 5|V'|}{3} - \frac{10}{3} = \frac{5|V|}{3} - \frac{10}{3} \geq \frac{5|V'|}{3} + |E \setminus E'| \geq \frac{5|E'| + 1}{3} + |E \setminus E'| \geq |E|.$$ 

This is exactly what we wanted to prove.

**Exercise 4 (Path removal)**

**Part (i)**

Let $P = v_1, v_2, \ldots, v_\ell$ be the longest path and assume $\ell \leq k$. As $\deg(v_\ell) \geq k$ there exists a vertex $w \in N(v_\ell) \setminus \{v_1, \ldots, v_{\ell-1}\}$. Thus $Pw$ forms a longer path than $P$ which is a contradiction. Therefore we have $\ell > k$.

**Part (ii)**

Assume $G[V \setminus \{v_1, \ldots, v_k\}]$ is disconnected. Let $K$ be a component of $G[V \setminus \{v_1, \ldots, v_k\}]$ which doesn’t contain vertices $v_{k+1}, \ldots, v_\ell$. Such $K$ exists as vertices $v_{k+1}, \ldots, v_\ell$ are contained in one component in $G[V \setminus \{v_1, \ldots, v_k\}]$. Furthermore, since $G$ is connected there exists a vertex $x \in K$ such that $\{x, v_i\} \in E$ for some $i \in [k]$. Let $Q$ be a maximal path in $K$ with $x$ as an endpoint and let us denote by $y$ the other endpoint of $Q$. If $Q$ is of length at least $k$ then path $v_\ell \ldots v_k \ldots v_iQ$ is longer than $P$ which is not possible. Therefore the path $Q$ is of length $t < k$. If $\deg_{G'[K]}(y) \geq t + 1$ then by the same argument as in Part (i), we could extend the path $Q$ which contradicts its maximality. Therefore we have that $\deg_{G'[K]}(y) \leq t < k$. On the other hand, as $\delta(G) \geq k$ that means
that by removing set \( \{v_1, \ldots, v_k\} \) we have deleted at least \( k - t \) edges from \( y \) to \( \{v_1, \ldots, v_k\} \). Let \( v_j \in \{v_1, \ldots, v_k\} \) be a vertex with smallest index such that \( \{y, v_j\} \in E \). By the previous argument we know that \( j \leq t + 1 \). However the path \( Qv_j \ldots v_k \ldots v_\ell \) is of length at least

\[
\ell - j + t + 1 \geq \ell - (t + 1) + t + 1 = \ell,
\]

which is a path longer than \( P \).