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Handout 1

Randomized Algorithms and Probabilistic Methods: Advanced Topics

You are encouraged to hand in your solutions to get feedback!

Exercise 1

In this exercise we want to recall how to approximate the sum of a monotone function by an appropriate integral. Prove the following statements.

(a) If $a \leq b$ are integers and $f\colon \mathbb{R} \to \mathbb{R}_0^+$ is monotone and integrable over $[a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x + \min\{f(a), f(b)\} \leq \sum_{k=a}^{b} f(k) \leq \int_a^b f(x)\,\mathrm{d}x + \max\{f(a), f(b)\}.$$

(b) $\sum_{k=1}^{n} \frac{1}{k} = \ln n + \mathcal{O}(1)$.

(c) $n! = \Theta(\sqrt{n}\left(\frac{n}{e}\right)^n)$.

Exercise 2

Consider the Trophy Collector Problem (we adopt the notation from the script). Show that for $a \geq 3$ the drift of Y_t is at least 0.5.

Exercise 3

Prove the following statement from the proof of the Multiplicative Drift Theorem (again, we adopt the notation from the script):

$$\mathbb{E}[X_\tau] \leq (1 - \delta)^\tau s_0.$$

Exercise 4

Let $(X_t)_{t\geq 0}$ be a time-homogeneous Markov chain with state space $0 \in S \subseteq \mathbb{R}_0^+$ and let T be the random variable that denotes the earliest point in time $t \geq 0$ such that $X_t = 0$. Assume that for each $x \in S$, the expectation $Y(x) = \mathbb{E}[T \mid X_0 = x]$ is finite. Show that the transformed random variables $Y_t := Y(X_t)$ have drift exactly 1.

Discussion of this handout on February 19th, 2015.