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Handout 3

Randomized Algorithms and Probabilistic Methods: Advanced Topics

Exercise 1

Let μ_1 and μ_2 be two distributions over the same finite set X. The total variation distance between them is

$$d_{TV}(\mu_1, \mu_2) := \frac{1}{2} \sum_{x \in X} |\mu_1(x) - \mu_2(x)|.$$

Show

$$d_{TV}(\mu_1, \mu_2) = \max_{X' \subseteq X} |\mu_1(X') - \mu_2(X')|,$$

with $\mu_i(X') := \sum_{x \in X'} \mu_i(x)$ for $X' \subseteq X$ and $i \in \{1, 2\}$.

Exercise 2

Let $G = (V, E)$ be a graph (undirected) with $n > 0$ vertices and consider a random walk on G (i.e., at each time step he choses a neighbor of its current vertex and walks there).

(a) Model the random walk as a finite and time-homogenous Markov chain $(X_t)_{t \geq 0}$ and determine the transition matrix in terms of the degrees and the adjacency matrix of G.

(b) Find a stationary distribution π. Is the stationary distribution unique?

(c) Does the distribution of the random walker converge to the stationary distribution?

Exercise 3

Let $G = (V, E)$ be a directed cycle with $n \geq 3$ vertices and self-loops on each vertex (i.e., $V = [n]$ and $E = \{(u, v) \in V^2 \mid v = u + 1 \mod n\} \cup \{(u, v) \in V^2 \mid u = v\}$) and consider a random walk on G.

(a) Show that the random walk corresponds to a finite, time-homogeneous, irreducible, and aperiodic Markov chain $(X_t)_{t \geq 0}$, and determine the stationary distribution π.

(b) Show that the mixing time is in $\mathcal{O}(n^2 \log n)$.