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Handout 4

Randomized Algorithms and Probabilistic Methods: Advanced Topics

Exercise 1

Recall that for a graph $G = (V, E)$ the edge expansion of G is defined as

$$h(G) = \min_{S \subseteq V, |S| \leq \frac{|V|}{2}} \frac{E(S, \bar{S})}{|S|}.$$

Show that for every $0 < \delta \leq 1$ the edge expansion of the random graph $G_{n,p}$ with $p \in \omega(\log n/n)$ a.a.s. satisfies

$$h(G_{n,p}) \geq \frac{(1-\delta)np}{2}.$$

Exercise 2

Consider the graph $G = (V, E)$ where the vertices can be partitioned into $k \geq 2$ sets of size $n \geq 1$, this is $V = V_1 \cup \ldots \cup V_k$ with $|V_i| = n$ for $1 \leq i \leq k$, and the edges are $E = \{\{u, v\} \mid u \in V_i \wedge v \in V_{i+1} \text{ with } 1 \leq i < k\} \cup \{\{u, u\} \mid u \in V_1 \vee u \in V_k\}$. Consider the following random walk on G: if the random walker is in a vertex in $V_2 \cup \ldots \cup V_{k-1}$, then he chooses its destination among all neighbors uniformely at random and if he is in $V_1 \cup V_k$, then he does not move with probability $1/2$ and otherwise choses its destination among all neighbors uniformely at random. Compute the stationary distribution π, show that the random walk converges to π and show

$$t_{mix} \leq \mathcal{O}(k^2 \log(nk))$$

using the flow method.

Exercise 3

Compute the eigenvalues of the hypercube.

Exercise 4

Let G be a (undirected) graph. Consider the random walk on G and let P be the corresponding transition matrix.

(a) Show that G has a bipartite component iff -1 is an eigenvalue of P.

(b) Show that the number of connected components of G is equal to the multiplicity of the eigenvalue 1 of P.

You can use the following theorem:

Theorem 1 (Perron-Frobenius)**.** *Any irreducible, stochastic matrix P has eigenvalue 1 with multiplicity 1 and the corresponding eigenvector is positive.*