ETH Zurich
Institute of Theoretical Computer Science
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FS 2015
Handout GHW 1

Randomized Algorithms and Probabilistic Methods: Advanced Topics

This is the first graded homework exercise set.

Regulations

- There will be a total of *three* special exercise sets during this semester.
- You are expected to solve them carefully and then write a nice and complete exposition of your solutions using LaTeX. Please submit your solutions as a pdf or ps file to `frank.mousset@inf.ethz.ch` (filename: [nethz user name]_ghw1.pdf/ps).
- You are welcome to discuss the exercises with your colleagues, but we expect each of you to hand in your own, *individual writeup.*
- Your solutions will be graded. Each solution will account for 10% of your final grade for the course (so 30% of the grade in total).

Due date: Wednesday, March 11th, 2015 at 23:59

Exercise 1

Consider the following game played with an even number $2n$ of coloured balls. The game proceeds in rounds. Initially, all balls are red. From then on, in every round, you must choose a ball uniformly at random and switch its colour (i.e., from red to blue and from blue to red). The game ends when there are as many red balls as blue ones.
Let T denote the number of rounds until the game ends. Show that

$$\mathbb{E}[T] \leq n \ln n + n.$$

Moreover, show that for every $c \in \mathbb{R}$, the probability that $T > \lceil n \ln n + cn \rceil$ is at most e^{-c}. *(15 points)*

Exercise 2

The coupon company has started selling a new kind of coupons, each of which may have either one or two pictures on it. More formally, let there be n different pictures, which we represent by the numbers 1 through n. Then, whenever a coupon is sold, we first choose k u.a.r. from $\{1, 2\}$, and then let the coupon have k pictures on it, each of which is chosen u.a.r. from $\{1, \ldots, n\}$. Note that a coupon may have the same picture twice.
Alice wants to start collecting these coupons. However, being an experienced coupon collector, she wants to make it a little bit more interesting: she wants to have each picture in her book an *odd* number of times (which is harder than just getting every picture at least once). Her strategy is the following: whenever she gets a new coupon, she checks if the total number of pictures that she owns an odd number of times would increase by adding this coupon to her book; if so, she glues the coupon into her book, if not, she does not. Note that she never removes a coupon from her book, even if it would make sense (this is because after a coupon has been glued into the book, it cannot be easily removed).
Let T be the number of coupons that she has to buy until she owns every picture an odd number of times. Prove that

$$\mathbb{E}[T] = 2n \ln n + \mathcal{O}(n),$$

where n is the number of different pictures. *(20 points)*

Exercise 3

Consider the random decline process $(X_t)_{t\geq 0}$ with $a = e$, that is, we start with $X_0 = n$, where n is some non-zero positive integer, and for every $t \geq 1$, we draw X_t uniformly at random from $\{0, 1, \dots, \lfloor eX_{t-1} \rfloor\}$.
Let T denote the first point in time for which $X_T = 0$ (and $T = \infty$ if there is no such point). Our goal in this exercise is to show that $\mathbb{E}[T] = \Omega(\ln n \ln\ln n)$.
For this, let $g(x) = \ln x \ln\ln x$, and let Y_t be a truncation of $g(X_t)$, i.e., $Y_t = g(X_t)$ if $X_t \geq e$, and $Y_t = 0$ otherwise.

(a) Show that the drift

$$\Delta(a) = \mathbb{E}[Y_t - Y_{t+1} \mid Y_t = a]$$

is at most constant. *(10 points)*

(b) Assuming $\lim_{t\to\infty} \mathbb{E}[Y_t] = 0$, prove that $\mathbb{E}[T] = \Omega(\ln n \ln\ln n)$. *(5 points)*

(c) Show that (b) also holds if you only assume $\liminf_{t\to\infty} \mathbb{E}[Y_t] = 0$. *Hint:* you may need to inspect the proof of Theorem 1.1. *(5 points)*

(d) Assuming $\liminf_{t\to\infty} \mathbb{E}[Y_t] > 0$, show that $\mathbb{E}[T] = \infty$. *Hint:* express $\Pr[Y_t > 0]$ in terms of $\mathbb{E}[Y_t]$ and use $\mathbb{E}[T] = \sum_t \Pr[X_t > 0]$. *(5 points)*