ETH Zurich
Institute of Theoretical Computer Science
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FS 2015
Handout GHW 2

Randomized Algorithms and Probabilistic Methods: Advanced Topics

This is the second graded homework exercise set.

Regulations

- There will be a total of *three* special exercise sets during this semester.
- You are expected to solve them carefully and then write a nice and complete exposition of your solutions using LaTeX. Please submit your solutions as a pdf or ps file to `frank.mousset@inf.ethz.ch` (filename: [nethz user name]_ghw1.pdf/ps).
- You are welcome to discuss the exercises with your colleagues, but we expect each of you to hand in your own, *individual writeup*.
- Your solutions will be graded. Each solution will account for 10% of your final grade for the course (so 30% of the grade in total).

Due date: Thursday, April 23th, 2015 at 23:59

Exercise 1

Let k be a positive integer constant, and let X be a fixed set of size $n \geq 1$. Consider the random walk on the set of all k-subsets of X that is defined as follows. In a given step, let Y_t be the current set. Pick elements $y \in Y_t$ and of $x \in X \setminus Y_t$ uniformly at random. The next state is then given by $Y_{t+1} = \{x\} \cup Y_t \setminus \{y\}$. Use the canonical paths method to show $t_{mix} \in \mathcal{O}(\log n)$.

(10 points)

Exercise 2

Let k be a positive integer constant. In this exercise, we consider placements of k identically-looking kings on a $k \times n$ for $n \geq 1$ chessboard, where no two kings can be placed on the same tile. We now define a random walk on the set of all such placements.
In chess, kings can move one square in any direction, including diagonally. At each step we choose a king and a direction (there are 8 directions) uniformly at random, and if we can move the chosen king in the chosen direction we make the move, and if not we stay in the same place. Use the method of canonical paths to prove that $t_{mix} \in \mathcal{O}(n^2 \log n)$.
You can use that $t_{mix} \in \mathcal{O}(\log n \cdot h(G)^{-2})$ even though the transition graph is not regular.

(25 points)

turn page

Exercise 3

Let $k \geq 1$ be an odd integer constant and consider the following random walk on $\mathbb{Z}_k^n$ for $n \geq 1$. In each step, we pick a coordinate $i \in [n]$ and a sign $s \in \{-1, +1\}$ uniformely at random and move from $(x_1, \ldots, x_i, \ldots, x_n)$ to $(x_1, \ldots, x_i + s, \ldots, x_n)$, where addition is modulo k. Note that the random walk is not lazy.

(a) Is the Markov chain ergodic?

(5 points)

(b) Use the canonical paths method to bound the mixing time of this random walk.

(10 points)

(c) Bound the mixing time of this random walk by bounding the spectral gap of the transition matrix.
Hint: start by computing the eigenvalues of the random walk on $\mathbb{Z}_k^1$ and proceed inductively.

(10 points)