ETH Zurich
Institute of Theoretical Computer Science
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FS 2015
Handout GHW 3

Randomized Algorithms and Probabilistic Methods: Advanced Topics

This is the third graded homework exercise set.

Regulations

- There will be a total of *three* special exercise sets during this semester.
- You are expected to solve them carefully and then write a nice and complete exposition of your solutions using LaTeX. Please submit your solutions as a pdf or ps file to `frank.mousset@inf.ethz.ch` (filename: [nethz user name]_ghw3.pdf/ps).
- You are welcome to discuss the exercises with your colleagues, but we expect each of you to hand in your own, *individual writeup.*
- Your solutions will be graded. Each solution will account for 10% of your final grade for the course (so 30% of the grade in total).

Due date: Wednesday, May 20th, 2015 at 23:59

Exercise 1

Let $G = (V, E)$ be a d-regular expander graph on n vertices with spectral expansion λ, where $n \geq d \geq 1$. The *eccentricity* of a vertex v of G is defined as

$$\epsilon(v) := \max_{u \in V} \operatorname{dist}(u, v),$$

where $\operatorname{dist}(u, v)$ is the distance of u and v in G, i.e., the minimum number of edges on a path from u to v in G. Show that there exists $K \in \mathbb{N}$ (which may depend on n) such that for every $m > 0$

$$|\{v \in V : |\epsilon(v) - K| \geq m\}| \leq Cne^{-cm}$$

for some constants $C, c > 0$ depending only on d and λ. *(20 points)*

Exercise 2

Let G be a d-regular graph on n vertices with spectral expansion λ, and let T_d be the infinite d-regular tree. For a graph H and $\ell \in \mathbb{N}$, let $p_\ell(H)$ denote the probability that if we choose a random vertex v in H and do a random walk of length 2ℓ, we we end back at vertex v.

(1) Show that $p_\ell(G) \geq p_\ell(T_d) \geq C_\ell (d-1)^\ell / d^{2\ell}$, where C_ℓ is the ℓ-th Catalan number, which equals the number of properly parenthesized strings in $\{(,)\}^{2\ell}$.

(2) Show that $n \cdot p_\ell(G) \leq 1 + (n-1) \cdot (\lambda/d)^{2\ell}$.

(3) Use the fact that $C_\ell = \binom{2\ell}{\ell}/(1+\ell)$ to show that

$$\lambda \geq 2\sqrt{d-1} - o(1),$$

where the $o(1)$ term tends to zero as $n \to \infty$. *(20 points)*

Exercise 3

Let $G = (V, E)$ be a d-regular graph on n vertices for some $n > d \geq 1$. Fix a non-empty set $S \subseteq V$ and define the vector $v \in \mathbb{R}^n$ by

$$v_i := \begin{cases} |\overline{S}| & \text{if } i \in S \\ -|S| & \text{otherwise.} \end{cases}$$

Show first that

$$\phi(S) = d - \frac{v^T A v}{n \cdot |S| \cdot |\overline{S}|},$$

where A denotes the adjacency matrix of G and where $\phi(S) = n|E(S, \overline{S})|/(|S||\overline{S}|)$. Then use this to show the easy direction of the proof of the Cheeger inequality:

$$d - \lambda_2 \leq \phi(S).$$

(20 points)