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Randomized Algorithms and Probabilistic Methods: Advanced Topics

Solution to Exercise 1

(a) Assume f is increasing and integrable on $[a, b]$. We can write

$$\sum_{k=a}^{b} f(k) \leq \sum_{k=a}^{b-1} \int_k^{k+1} f(x)\,\mathrm{d}x + f(b) = \int_a^b f(x)\,\mathrm{d}x + f(b),$$

since $f(k) \cdot 1 \leq \int_k^{k+1} f(x)\,\mathrm{d}x$ for increasing f. Furthermore,

$$\sum_{k=a}^{b} f(k) \geq \sum_{k=a+1}^{b} \int_{k-1}^{k} f(x)\,\mathrm{d}x + f(a) = \int_a^b f(x)\,\mathrm{d}x + f(a),$$

since $f(k) \cdot 1 \geq \int_{k-1}^{k} f(x)\,\mathrm{d}x$ for increasing f. The proof for decreasing f is analogous and the statement follows.

(b) Let $H_n := \sum_{k=1}^{n} 1/k$. We can simply apply the result from (a) to get

$$\ln n + \frac{1}{n} = \int_1^n \frac{1}{x}\,\mathrm{d}x + \frac{1}{n} \leq H_n \leq \int_1^n \frac{1}{x}\,\mathrm{d}x + 1 = \ln n + 1.$$

Looking closely at the error term in (a), we can actually obtain the stronger statement that the limit $\gamma = \lim_{n\to\infty} H_n - \ln n$ exists. The constant γ is usually called the Euler-Mascheroni constant.

(c) Write

$$\ln n! = \sum_{k=1}^{n} \ln k = \ln n + \int_1^n \ln x\,\mathrm{d}x - \sum_{k=1}^{n-1} \int_k^{k+1} \ln x\,\mathrm{d}x - \ln k.$$

Recall that $\int_a^b \ln x\,\mathrm{d}x = b\ln b - b - a\ln a + a$. If we write $\Delta_k := \int_k^{k+1} \ln x\,\mathrm{d}x - \ln k$, then this simplifies to

$$\ln n! = \ln n + n\ln n - n + 1 - \sum_{k=1}^{n-1} \Delta_k. \tag{1}$$

We get

$$\begin{aligned}
\sum_{k=1}^{n-1} \Delta_k &= \sum_{k=1}^{n-1} (k+1)\ln(k+1) - k\ln k - \ln k - 1 \\
&= \sum_{k=1}^{n-1} (k+1)(\ln(k+1) - \ln k) - 1 \\
&= \sum_{k=1}^{n-1} (k+1)\ln(1+1/k) - 1 \\
&= \sum_{k=1}^{n-1} (k+1)\left(\frac{1}{k} - \frac{1}{2k^2} + \mathcal{O}(k^{-3})\right) - 1 && (2) \\
&= \sum_{k=1}^{n-1} \frac{1}{2k} + \mathcal{O}(k^{-2}) && (3) \\
&= \frac{1}{2}\ln n + \mathcal{O}(1), && (4)
\end{aligned}$$

where in (2) we applied Taylor's theorem, in (3) we applied (b), and in the last line, we used the well-known fact that $\sum_{k=1}^{\infty} 1/k^2$ converges. Plugging this into (1), we get

$$\ln n! = n \ln n - n + 1 + \frac{\ln n}{2} + \mathcal{O}(1),$$

so

$$n! = e^{\mathcal{O}(1)} \cdot \sqrt{n} \left(\frac{n}{e}\right)^n,$$

as required. Again, by doing the same proof a little more carefully, we obtain that there exists a constant c such that $n! \sim c \cdot \sqrt{n}(n/e)^n$. Surprisingly, it turns out that $c = \sqrt{2\pi}$. This result is usually referred to as Stirling's approximation to the factorial.

Solution to Exercise 2

For $a \leq 2$ we see that $\Delta(a) \leq 0.5$ because the number of missing trophies cannot be negative. Hence, for $a = 2$ we get $X_{t+1} = 0$ if the team wins 2 or 3 trophies and for $a = 1$ we get $X_{t+1} = 0$ if the team wins 1, 2 or 3 trophies. Similarly, for $a \geq 3$ we have $\Delta'(a) \geq 0.5$: for $a = 3$, if the team wins at 1, 2 or 3 trophies, then Y_{t+1} is 0 and for $a = 4$, if the team wins 2 or 3 trophies, then Y_{t+1} is 0. In both cases, $Y_t - Y_{t+1} \geq 3$.

Solution to Exercise 3

Equation (1.2) from the lecture notes is equivalent to

$$\mathbb{E}[X_{\tau+1} \mid X_\tau] \leq (1-\delta)\mathbb{E}[X_\tau \mid X_\tau] = (1-\delta)X_\tau,$$

by linearity of expectation. Taking the expectation on both sides,

$$\mathbb{E}[\mathbb{E}[X_{\tau+1} \mid X_\tau]] \leq (1-\delta)\mathbb{E}[X_\tau].$$

Since $\mathbb{E}[\mathbb{E}[X_{\tau+1} \mid X_\tau]] = \mathbb{E}[X_{\tau+1}]$, we get $\mathbb{E}[X_{\tau+1}] \leq (1-\delta)\mathbb{E}[X_\tau]$, and so, by induction, we obtain

$$\mathbb{E}[X_\tau] \leq (1-\delta)^\tau \mathbb{E}[X_0] = (1-\delta)^\tau s_0,$$

as required.

Solution to Exercise 4

Let $(X_t)_{t\geq 0}$ be a (time-homogeneous) Markov chain with state space $0 \in S \subseteq \mathbb{R}_0^+$ such that the values $Y(x) = \mathbb{E}[T \mid X_0 = x]$ are well-defined. Note that $Y(a) = 0$ if and only if $a = 0$.

For all $x_0 \neq 0$, we have

$$\begin{aligned} Y(x_0) = \mathbb{E}[T \mid X_0 = x_0] &= \sum_{x_1 \in S} \Pr[X_1 = x_1 \mid X_0 = x_0] \cdot \mathbb{E}[T \mid X_1 = x_1] \\ &= \sum_{x_1 \in S} \Pr[X_1 = x_1 \mid X_0 = x_0] \cdot (1 + \mathbb{E}[T \mid X_0 = x_1]) \\ &= 1 + \mathbb{E}[Y_1 \mid X_0 = x_0]. \end{aligned}$$

Thus, for all $a \neq 0$, we have

$$\mathbb{E}[Y_{t+1} \mid Y_t = a] = \mathbb{E}[Y_1 \mid Y_0 = a] = \sum_{x \in S} \Pr[X_0 = x \mid Y_0 = a] \cdot \mathbb{E}[Y_1 \mid X_0 = x] = a - 1,$$

so

$$\mathbb{E}[Y_t - Y_{t+1} \mid Y_t = a] = 1,$$

as required.