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Solution 1

Randomized Algorithms and Probabilistic Methods: Advanced Topics

Solution to Exercise 1

(a) First, we show that the Markov property implies $\mathbb{E}[T \mid X_1 = x_1 \wedge X_0 = x_0] = \mathbb{E}[T \mid X_1 = x_1]$, for $x_0 \neq 0$:

$$
\begin{aligned}
&\mathbb{E}[T \mid X_1 = x_1 \wedge X_0 = x_0] = \\
&\quad \sum_{t \geq 0} t \cdot \Pr[T = t \mid X_1 = x_1 \wedge X_0 = x_0] = \\
&\quad \sum_{t \geq 0} t \cdot \sum_{x_{t-1},\dots,x_2} \Pr[X_t = 0 \wedge X_{t-1} = x_{t-1} \wedge \ldots \wedge X_2 = x_2 \mid X_1 = x_1 \wedge X_0 = x_0] = \\
&\quad \sum_{t \geq 0} t \cdot \sum_{x_{t-1},\dots,x_2} \Pr[X_t = 0 \mid X_{t-1} = x_{t-1} \wedge \cdots \wedge X_0 = x_0] \cdots \Pr[X_2 = x_2 \mid X_1 = x_1 \wedge X_0 = x_0] = \\
&\quad \sum_{t \geq 0} t \cdot \sum_{x_{t-1},\dots,x_2} \Pr[X_t = 0 \mid X_{t-1} = x_{t-1}] \cdots \Pr[X_2 = x_2 \mid X_1 = x_1] = \\
&\quad \sum_{t \geq 0} t \cdot \sum_{x_{t-1},\dots,x_2} \Pr[X_t = 0 \wedge X_{t-1} = x_{t-1} \wedge \cdots \wedge X_2 = x_2 \mid X_1 = x_1] = \\
&\quad \sum_{t \geq 0} t \cdot \Pr[T = t \mid X_1 = x_1] = \\
&\quad \mathbb{E}[T \mid X_1 = x_1],
\end{aligned}
$$

where the sum $\sum_{x_{t-1},\dots,x_2}$ is over the elements of $(S \setminus 0)^{t-2}$. Hence, we can conclude

$$
\begin{aligned}
&\sum_{x_1 \in S} \Pr[X_1 = x_1 \mid X_0 = x_0] \cdot \mathbb{E}[T \mid X_1 = x_1] = \\
&\quad \sum_{x_1 \in S} \Pr[X_1 = x_1 \mid X_0 = x_0] \cdot \mathbb{E}[T \mid X_1 = x_1 \wedge X_0 = x_0] = \\
&\quad \sum_{x_1 \in S} \Pr[X_1 = x_1 \mid X_0 = x_0] \cdot \sum_{t \geq 0} t \cdot \Pr[T = t \mid X_1 = x_1 \wedge X_0 = x_0] = \\
&\quad \sum_{x_1 \in S} \sum_{t \geq 0} t \cdot \Pr[X_1 = x_1 \mid X_0 = x_0] \cdot \Pr[T = t \mid X_1 = x_1 \wedge X_0 = x_0] = \\
&\quad \sum_{x_1 \in S} \sum_{t \geq 0} t \cdot \Pr[T = t \wedge X_1 = x_1 \mid X_0 = x_0] = \\
&\quad \sum_{t \geq 0} t \cdot \sum_{x_1 \in S} \Pr[T = t \wedge X_1 = x_1 \mid X_0 = x_0] = \\
&\quad \sum_{t \geq 0} t \cdot \Pr[T = t \mid X_0 = x_0] = \\
&\quad \mathbb{E}[T \mid X_0 = x_0].
\end{aligned}
$$

(b) We simply calculate

$$\begin{aligned}\mathbb{E}[\mathbb{E}[X \mid Y]] &= \sum_y \mathbb{E}[X \mid Y = y] \cdot \Pr[Y = y] \\ &= \sum_y \sum_x x \cdot \Pr[X = x \mid Y = y] \cdot \Pr[Y = y] \\ &= \sum_y \sum_x x \cdot \Pr[X = x \wedge Y = y] \\ &= \sum_x \sum_y x \cdot \Pr[X = x \wedge Y = y] \\ &= \sum_x x \cdot \Pr[X = x] \\ &= \mathbb{E}[X].\end{aligned}$$

Solution to Exercise 2

Let X_t be the random variable counting the missing coupons at time t and T be the first point in time t such that $X_t = 0$. First, note that $\mathbb{E}[T \mid X_0 = n] \geq (1 - o(1))n \ln n$ because the process is at most as fast as the Coupon Collector process.
Hence, it is left to show an upper bound. Let $\varepsilon > 0$ and $T_{\varepsilon n}$ be the first point in time t such that $X_t = \varepsilon n$. We can write $\mathbb{E}[T \mid X_0 = n] = \mathbb{E}[T_{\varepsilon n} \mid X_0 = n] + \mathbb{E}[T \mid X_0 = \varepsilon n]$.

To compute $\mathbb{E}[T_{\varepsilon n} \mid X_0 = n]$ define the random variables Y_t as

$$Y_t := \begin{cases} X_t, & \text{if } X_t > \varepsilon n, \\ 0, & \text{otherwise,} \end{cases}$$

and denote by T_Y the first point in time t such that $Y_t = 0$. We apply the Multiplicative Drift theorem with $\delta = 1/(2n)$, $s_{min} = \varepsilon n$, and $s_0 = n$:

$$\mathbb{E}[T_{\varepsilon n} \mid X_0 = n] = \mathbb{E}[T_Y \mid Y_0 = n] \leq \frac{1 + \ln \frac{1}{\varepsilon}}{\frac{1}{2n}} = \mathcal{O}(n).$$

Now we determine $\mathbb{E}[T \mid X_0 = \varepsilon n]$. Let $\mathcal{E}$ be the event that Alice loses all coupons. We have

$$\begin{aligned}\mathbb{E}[T \mid X_0 = \varepsilon n] &= (1 - \frac{1}{2n})^{\varepsilon n} \cdot \mathbb{E}[T \mid X_0 = \varepsilon n \wedge \overline{\mathcal{E}}] + (1 - (1 - \frac{1}{2n})^{\varepsilon n}) \cdot \mathbb{E}[T \mid X_0 = \varepsilon n \wedge \mathcal{E}] \\ &\leq \mathbb{E}[T \mid X_0 = \varepsilon n \wedge \overline{\mathcal{E}}] + \frac{\varepsilon}{2} \cdot \mathbb{E}[T \mid X_0 = \varepsilon n \wedge \overline{\mathcal{E}}],\end{aligned}$$

where we used Bernoulli's inequality in the last line. Note that $\mathbb{E}[T \mid X_0 = \varepsilon n \wedge \overline{\mathcal{E}}] = (1 + o(1))n \ln n$, since conditioned on the event $\overline{\mathcal{E}}$ the process behaves like the Coupon Collector process. Furthermore, we can bound $\mathbb{E}[T \mid X_0 = \varepsilon n \wedge \mathcal{E}] \leq 2\mathbb{E}[T \mid X_0 = n]$. Solving yields $\mathbb{E}[T \mid X_0 = \varepsilon n] \leq (1 + o(1))n \ln n$.

Solution to Exercise 3

We consider the case $a \leq 1$. Define the random variables Y_t as

$$Y_t := \begin{cases} \ln X_t, & \text{if } X_t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that Y_t is 0 if X_t is 0 or 1. Hence, to show the lower bound for the Random Decline process, it suffices to show $\mathbb{E}[T \mid Y_0 = \ln n] = \Omega(\ln n)$. The drift of $(Y_t)_{t \geq 0}$ is

$$\begin{aligned}\mathbb{E}[Y_t - Y_{t+1} \mid Y_t = \ln x] &= \ln x - \sum_{i=2}^{\lfloor ax \rfloor} \frac{\ln i}{1 + \lfloor ax \rfloor} \\ &\leq \ln x - \frac{1}{1 + \lfloor ax \rfloor} \cdot (\lfloor ax \rfloor \ln \lfloor ax \rfloor - \lfloor ax \rfloor) \\ &= \mathcal{O}(1).\end{aligned}$$

Thus, we can apply the Additive Drift theorem to obtain that $\mathbb{E}[T \mid Y_0 = \ln n] = \Omega(\ln n)$. For $a > 1$ note that we can couple the processes: Let $(X_t)_{t\geq 0}$ be the process with $a > 1$ and $(X'_t)_{t\geq 0}$ be the process with $a = 1$. The coupling is as follows: if $X_{t+1} \leq X'_t$ we set $X'_{t+1} = X_{t+1}$ and otherwise $(X'_t)_{t\geq 0}$ does a regular step. Hence, $X_t \geq X'_t$ holds for all $t \geq 0$ and $(X_t)_{t\geq 0}$ can not be faster than $(X'_t)_{t\geq 0}$.

Solution to Exercise 4

(i) If there is a weight $w_i < 0$, then we can just apply the automorphism φ to the hypercube that flips 0 and 1 in the i-th bit, and replace the weight $w_i < 0$ by $w'_i := -w_i > 0$.

Note that then we have $f_w(x) - f_w(y) = f_{w'}(\varphi(x)) - f_{w'}(\varphi(y))$ for all $x, y \in H_n$. Therefore, by setting $y = 0$, we get $f_w(x) = f_{w'}(\varphi(x)) - f_{w'}(\varphi(0))$ for all $x \in H_n$, so both problems are equivalent. Not only do they take the miminum in the same places, have the same optimum, but the optimization algorithm makes the same choices on both instances, modulo φ.

Similarly, changing the order of the bits does not change the problem.

(ii) The upper bound is trivial. For the lower bound, observe that for fixed i, the probability that we flip exactly the i-th bit is $1/n \cdot (1 - 1/n)^{n-1}$. We use the well-known formula $(1 - x/n)^n \leq e^{-x} \leq (1 - x/n)^{n-1}$ for $x = 1$. (It holds for all $x \geq 0$.) Now we sum over all i and obtain

$$p_0 = \left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{1}{e}.$$

(iii) For $a = 0$ the statement is trivial. For $a > 0$, we compute directly

$$\begin{aligned}\mathbb{E}[X_t - X_{t+1} \mid X_t = a] &\geq \mathbb{E}[X_t - X_{t+1} \mid X_t = a \text{ and we flip exactly one bit.}] \cdot \frac{1}{e} \\ &= \frac{1}{e} \cdot \sum_{i\text{-th bit wrong}} \Pr[\text{flip exactly } i\text{-th bit}] \cdot w_i \\ &\geq \frac{1}{en} \cdot \sum_{i\text{-th bit wrong}} w_i \\ &= \frac{a}{en}.\end{aligned}$$

(iv) We apply the Variable Drift theorem, where $s_{\min} = w_{\min}$ and $h(x) = x/(en)$. We obtain

$$\mathbb{E}[T \mid X_0 = a] \leq \frac{w_{\min}}{h(w_{\min})} + \int_{w_{\min}}^{a} \frac{1}{h(x)} dx = en \left(1 + \int_{w_{\min}}^{a} \frac{1}{x} dx\right) = en \left(1 + \log\left(\frac{a}{w_{\min}}\right)\right).$$

Observing that the starting value is $a = \sum_{i=1}^{n} w_i$ yields the result.