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Randomized Algorithms and Probabilistic Methods: Advanced Topics

Solution to Exercise 1

A proof can be found in the lecture notes of Randomized Algorithms and Probabilistic Methods.

Solution to Exercise 2

(a) Let A_G be the adjacency matrix of G, this is $A_G \in \{0,1\}^{n\times n}$ with $a_{i,j} = 1$ iff $\{i,j\} \in E$ (we identify the vertices $V(G)$ with $[n]$). Let D be the diagonal matrix with $d_{i,i} = 1/\deg(i)$. Then the transition matrix is $P = A_G \cdot D$.

(b) The stationary distribution is $\pi = (\deg(1)/n, \dots, \deg(n)/n)$. It is easy to see that $\pi P = \pi$:

$$(\pi P)_j = \sum_{i=0}^{n} \pi_i \cdot p_{i,j} = \frac{1}{2n} \sum_{\substack{i=0 \\ p_{i,j}\neq 0}}^{n} \deg(i) \cdot \frac{1}{\deg(i)} = \pi_j.$$

If G is connected, then the stationary distribution is unique (the Markov chain is irreducible). Otherwise it is not: any convex combination of stationary distributions of random walks on the connected components is also a stationary distribution.

(c) If G is not bipartite, then the distribution of the random walker converges to the stationary distribution (the Markov chain is aperiodic). Otherwise it does not: if the random walker starts on the left partition of G, then after an even number of steps, he will always be on the left partition.

Solution to Exercise 3

(a) Let A_G be the adjacency matrix of G, this is $A_G \in \{0,1\}^{n\times n}$ with $a_{i,j} = 1$ iff $(i,j) \in E$. The transition matrix is $P = 1/2A_G$. The Markoc chain is irreducible since G is strongly connected and aperiodic because of the self-loops. The stationary distribution is $\pi = (1_n, \dots, 1/n)$ by symmetry.

(b) We want to apply Theorem 2.2 from the lecture. First, we determine λ, the second largest eigenvalue of P (in absolute value). To do so write $P = 1/2(I + M)$ and note that if λ_M is an eigenvalue of M, then $\lambda_P = (\lambda_M + 1)/2$ is an eigenvalue of P. To find the eigenvalues of M we must determine the roots of the characteristic polynomial $\det(M - Ix) = x^n - 1$. Hence, we see that the eigenvalues of M are the roots of unity $e^{2\pi i \frac{k}{n}}$ for $0 \le k \le n$. Thus, the eigenvalues of P are $(e^{2\pi i \frac{k}{n}} + 1)/2$ for $0 \le k \le n$. We find that $\lambda = (e^{2\pi/n} + 1)/2$ and $|\lambda| = \sqrt{1/2 + \cos(2\pi/n)/2}$. Using a Taylor approximation we see that $\cos(2\pi/n) \le 1 - 2/n^2$ and therefore $|\lambda| \le e^{-1/(2n^2)}$.
Second we need to show that the eigenvectors of P are orthogonal. Using the spectral theorem it suffices to show that P is normal, this is $P^T \cdot P = P \cdot P^T$. This holds iff M is normal and we see that this holds since $M^T \cdot M = M^T \cdot M = I$.
Finally, we can apply Theorem 2.2 to find that

$$d(t) \le \sqrt{n} e^{-t/(2n^2)}.$$

We need that $\sqrt{n} e^{-t/(2n^2)} \le 1/4$ which is achieved by $t = \lceil n^2 \ln n + 2n^2 \ln 4 \rceil$.