ETH Zurich
Institute of Theoretical Computer Science
Dr. Karl Bringmann, Dr. Johannes Lengler, and Dr. Tsur Luria
Frank Mousset, Felix Weissenberger

Randomized Algorithms and Probabilistic Methods: Advanced Topics

Solution to Exercise 1

Fix $0 \leq \delta < 1$. For a set $S \subseteq V$ with $|S| \leq |V|/2$ denote by $\mathcal{E}_S$ the event that $E(S, \overline{S}) \in (1 \pm \delta)|S||\overline{S}|p$ holds. We apply the Chernoff bound to get

$$\Pr[\mathcal{E}_S] \geq 1 - 2e^{-\delta^2 |S||\overline{S}|p/3}.$$

Now we use the the union bound over those sets S to get that for n large enough

$$\begin{aligned}
\Pr[\exists S \subseteq V \text{ with } |S| \leq |V|/2 \text{ and } \overline{\mathcal{E}}_S] &\leq \sum_{k=0}^{n/2} \binom{n}{k} 2e^{-\delta^2 k(n-k)p/3} \\
&\leq 2 \sum_{k=0}^{n/2} n^k e^{-\delta^2 knp/6} \\
&\leq 2 \sum_{k=0}^{\infty} e^{-k(\delta^2 np/6 - \log n)} \\
&\leq 2\left(\frac{1}{1 - e^{-\delta^2 np/6 + \log n}} - 1\right) \overset{n\to\infty}{\to} 0
\end{aligned}$$

holds, since $p \in \omega(\log n/n)$. Hence, we get that a.a.s. the edge expansion is

$$h(G_{n,p}) \geq \frac{(1-\delta)np}{2}.$$

Solution to Exercise 2

The stationary distribution is uniform: $\pi_i = 1/(kn)$ for $i \in V$. The Markov chain is connected and therefore irreducible, moreover since it has self loops and is undirected it is also aperiodic. Hence, it converges to π. To apply the flow method, we have to route one unit of flow, for each pair (u, v) of vertices. We do this as follows: if u and v are connected via an edge e, then we route the unit corresponding to (u, v) directly over e. Otherwise, we distribute the flow over all shortes paths from u to v. Now we want to bound the amount of flow on an edge $e = \{u, v\}$, with $u \in V_i$ and $v \in V_{i+1}$ for $1 \leq i < k$. We get

$$f(e) = 2 + 2\frac{(i-1)n \cdot (k-(i+1))n}{n^2} + 2kn/n,$$

since the endpoints send two units, the pairs involving only one of the enpoints (at most $2kn$ many) spread their flow over at least n paths, and the $(i-1)n \cdot (k-(i+1))n$ remaining pairs send 2 units which are spread out among the n^2 possible edges between V_i and V_{i+1}. We see that $f(e)$ is maximized for $i - 1 = k/2$ and we can bound it by $f(e) \leq k^2/2$, at least if k is large. This yields $h(G) \geq n/k$ by Theorem 2.7 and therefore yields

$$t_{mix} \leq 16k^2 \log(2\sqrt{nk})$$

Solution to Exercise 3

We denote by A_n the adjacency matrix of the hypercube of dimension n. Recall that we can construct the n-th hypercube by taking two copies of the hypercube of dimension $n-1$ and putting edges between vertices with the same label. Hence, we can also construct the adjacency matrix inductively as follows:

$$A_n = \begin{pmatrix} A_{n-1} & I_{n-1} \\ I_{n-1} & A_{n-1} \end{pmatrix} \text{ for } n > 1 \text{ and } A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where here I_{n-1} is the identity of dimension 2^{n-1}. We will see that we can also construct the eigenvalues recursively. It is easy to see that the eigenvalues of A_1 are 1 and -1. Using the recursive definition of A_n, we see that if v is an eigenvalue of A_{n-1} with eigenvalue λ, then (v, v) and $(v, -v)$ are eigenvectors of A_n with eigenvalues $\lambda + 1$ and $\lambda - 1$. Moreover, it is easy to see that the eigenvectors constructed in this way are linearly independent. Hence, the eigenvalues of A_n are $n, n-2, \ldots, -n+2, -n$. If we also want to count the multiplicity we see that the multiplicity of the k-th eigenvalue of A_n is given as the sum of the multiplicities of the $k-1$-th and k-th eigenvalues of A_{n-1}. Hence, the multiplicity of the k-th eigenvalue of A_n satisfies the recursen of the binomial coefficient and plugging in the start values we get that it is $\binom{n}{k}$.

Solution to Exercise 4

(a) We assume that G is connected (an eigenvector of a connected componend yields an eigenvector of P by padding zeros and the corresponding eigenvalue is the same). First, we show that if G is bipartite, then P has eigenvalue -1. Let V_1 and V_2 be the partitions of the vertex set. We define $\overline{\pi}$ by $\overline{\pi}_v = \deg(v)/(2|E|)$ for $v \in V_1$ and $\overline{\pi}_v = -\deg(v)/(2|E|)$ for $v \in V_2$. Then we see for $v \in V_1$

$$(\overline{\pi}^T P)_v = \sum_{u \in V} \overline{\pi}_u P_{u,v} = \sum_{u | \{u,v\} \in E} -\frac{\deg u}{2|E|} \cdot \frac{1}{\deg(u)} = -\overline{\pi}_v$$

and analogously for $v \in V_2$. Hence, $\overline{\pi}$ is an egenvector with eigenvalue -1. Second, we show that if P has eigenvalue -1, then G is bipartite. Let $\overline{\pi}$ be an eigenvector of P with associated eigenvalue -1. By the Perron-Frobenius Theorem, P has a positive eigenvector π with associated egenvalue 1. We claim that $|\overline{\pi}|$ is an eigenvector with eigenvalue 1 as well, and therefore is a multiple of π and thus is positive as well. This can be seen since

$$|\overline{\pi}|^T = |-\overline{\pi}^T| = |\overline{\pi}^T P| \leq |\overline{\pi}^T| P$$

and multiplication with π shows that in fact equality holds, i.e., $|\overline{\pi}^T P| = |\overline{\pi}^T| P$. Now let V_1 be the set of vertices v with $\overline{\pi}_v > 0$ and V_2 be the set of vertices v with $\overline{\pi}_v < 0$. Fix $v \in V$ and note that the above claim implies

$$\left| \sum_{u \in V_1} |\overline{\pi}_u| P_{u,v} - \sum_{u \in V_2} |\overline{\pi}_u| P_{u,v} \right| = |(\overline{\pi}^T P)_v| = (|\overline{\pi}^T| P)_v = \sum_{u \in V_1} |\overline{\pi}_u| P_{u,v} + \sum_{u \in V_2} |\overline{\pi}_u| P_{u,v}.$$

Since $|\overline{\pi}| > 0$ and $P \geq 0$, we see that either $\sum_{u \in V_1} |\overline{\pi}_u| P_{u,v}$ or $\sum_{u \in V_2} |\overline{\pi}_u| P_{u,v}$ must be 0. However, this shows that v can either be connected to vertices from V_1 or V_2, bot not to both. If $v \in V_1$ and it is connected to only vertices in V_1, then there exists a connected component whose vertices are a subset of V_1 contradicting the assumption that the graph is connected. Hence, a vertex in V_1 can only be connected to vertices in V_2 and vice versa. But this shows that G is bipartite.

(b) For each component C_i with vertices V_i and endges E_i we define $\pi^{(i)}$ by $\pi^{(i)}_v = 0$ if $v \notin V_i$ and $\pi^{(i)}_v = \deg(v)/(|E_i|)$. Obviously, this yields an eigenvector with eigenvalue 1 for each component and those eigenvectors are pairwise orthogonal. Thus, the multiplicity of the eigenvalue 1 is at least the number of connected components of G. However, if there was an additional eigenvector with eigenvalue 1 which is linearly independent to all the $\pi^{(i)}$, then there must exist a component with two linearly independent eigenvalues with eigenvalue 1 contradicting the Perron-Frobenius Theorem.