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Solution 5

Randomized Algorithms and Probabilistic Methods: Advanced Topics

Solution to Exercise 1

(a) The row/clumn/symbol-swaps commute. Hence, there are at most $n!^3$ vertices reachable from each vertex. However, $n!^3 \ll ((1+o(1))(n/e^2))^{n^2}$.

(b) Consider an order-n cyclic latin square (the addition table of $\mathbb{Z}_n$), where n is prime. Fix two rows i_1 and i_2. Look at the induced permutation on the symbols: if we follow the permutation, there is an offset of $d := i_2 - i_1$. Going along a cycle of length l we have that $n|l \cdot d$. Since n is prime we have $n|l$ or $n|d$. However, $d < n$ and $l \leq n$ impliy that $l = n$ and the permutation has only one cycle. Swapping it corresponds to a row-swap. Thus, by (a) a component containing a cyclic Latin square has size at most $n!$.

(c) We consider an order-n cyclic latin square (the addition table of $\mathbb{Z}_n$), where n is prime an show that it does not contain a non-trivial Latin subsquare. This shows that row/column/symbol swaps in Latin subsquares corresponds to row/column/symbol swaps and therefore by (a) a component containing a cyclic Latin square has size at most $n!^3$. First, we show that if it contains a Latin subsquare, then n is not prime. Let $\{i_1, \dots, i_k\}$ be the row- and $\{j_1, \dots, j_k\}$ be the column-indices of an order-k Latin subsquare. We see that $\sum_{x=1}^{k} i_1 + j_x = \sum_{x=1}^{k} i_2 + j_x$ and therefore $ki_1 = ki_2$ which implies that $n|k \cdot (i_1 - i_2)$ and since n is prime $n|k$ or $n|(i_1 - i_2)$. However, since $i_1 - i_2 < n$ and $k \leq n$ we see that $k = n$ and the Latin subsquare is not a proper subsquare. Second, we show that if n is not prime, then it indeed contains a Latin subsquare. If n is not prime, then there exists a subgroup $H \subset \mathbb{Z}_n$. We show that this H forms a Latin subsquare. Since the H is closed, there are only the elements of H in the subsquare. Now assume that there exists a row (column) i and two columns (rows) j_1, j_2 in the subsquare such that $i + j_1 = i + j_2$. Since we look at a subgoup, we have $j_1 = j_2$ and the subsquare is Latin.