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Solution 6

Randomized Algorithms and Probabilistic Methods: Advanced Topics

Solution to Exercise 1

Let CH be the convex hull of Q_1. We show $K_1 \subseteq \bigcup_{p \in CH} B_2(p/2, ||p/2||)$. This means that if for each $p \in CH$ we place an Euclidean Ball of radius $||p/2||$ on the center of the line between the origin and p, then all of these balls together cover K_1. To see this fix some $p \in CH$ and a point $x \notin B_2(p/2, ||p/2||)$. Since x is not in $B_2(p/2, ||p/2||)$, we have that $||x - p/2|| > ||p/2||$ an therefore $\langle x, p \rangle < ||x||^2$. But this means, that p is in the open halfspace H bounded by the halfplane through x perpendicular to x. However, if x is not in any of the balls, then all $p \in CH$ are in H and therefore $CH \subseteq H$. But this shows that $x \notin CH$.
Now note that the infinity norm ball is a superset of the Euclidean norm ball. Thus

$$vol(K_1) \leq \bigcup_{p \in CH} B_2(p/2, ||p/2||) \leq \bigcup_{p \in CH} B(p/2, ||p/2||) = |CH| \cdot (2||p/2||)^n \leq poly(n)n^n.$$

On the other hand $nB \subseteq K_2$ and therefore $vol(K_2) \geq (2n)^n$ proving the claim.

Solution to Exercise 2

We describe an algorithm which computes the transformation f. Start with the right simplex $S := CH(0, e_1, \ldots, e_n)$, where CH denotes the convex hull and $(e_i)_{i=1}^n$ is the standard basis of $\mathbb{R}^n$. Note that $S \subseteq B \subseteq K$. Now while there exists a point $p \in K$ such that $p_i \geq (1 + 1/n^2)$ set $S' = CH(0, e_1, \ldots, p_i, \ldots, e_n)$. Observe that the volume of S' exceeds the volume of S by a factor of at least $(1 + 1/n^2)$. Then, rescale such that S' becomes the simplex again and note that thereby the volume of K shrinks by a factor of at least $(1 + 1/n^2)$ (this builds the transformation f, however for simplicity we always denote the *new* K again by K). Since $S \subseteq K$ during the entire process, the number of iterations is bounded by $poly(n)$. At the end of the algorithm we know $B' \subset S \subseteq K \subseteq (1 + 1/n^2)B$, where B' is the infinity norm ball with radius $1/(2n)$ centered at $1/(2n)\mathbf{1}$, where $\mathbf{1}$ is the all 1's vector. Now we rescale (and shift) such that B' becomes B. This yields $B \subseteq K \subseteq 2n(1 + 1/(2n) + 1/n^2)B \subseteq 2(n + 1)B$.