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FS 2015
Solution 7

Randomized Algorithms and Probabilistic Methods: Advanced Topics

Solution to Exercise 1

Let G be a d-regular expander graph on n vertices with spectral expansion $\lambda < d$. Let H be any (spanning) subgraph of G obtained by removing $m \geq 0$ edges. We will prove that H contains a component of size at least $n - 4m/(d-\lambda)$. We distinguish between two cases.

If H contains a component X of size at least $n/2$, then by the Cheeger inequality and the fact that X is a component of H, we have

$$2m \geq 2e_G(X, V(G) \setminus X) \geq (d-\lambda)(n - |X|).$$

Rearranging, we get $|X| \geq n - 2m/(d-\lambda)$, so X is the component of the desired size.

Now assume that all components of H have size smaller than $n/2$. Let $X_1, X_2, \ldots, X_k$ be the components of H. By the Cheeger inequality, we have

$$e_G(X_i, V(G) \setminus X_i) \geq (d-\lambda)|X_i|/2$$

for every $1 \leq i \leq k$. Since the sets X_i are components of H, we get

$$2m \geq \sum_{i=1}^{k} e_G(X_i, V(G) \setminus X_i) \geq \sum_{i=1}^{k} (d-\lambda)|X_i|/2 = (d-\lambda)n/2.$$

Then $m \geq (d-\lambda)n/4$, and H trivially contains a component of size $0 \geq n - 4m/(d-\lambda)$.

Solution to Exercise 2

Let G be a d-regular graph on n vertices with spectral expansion $\lambda < d$.

Fix any $v \in V(G)$. Suppose that, for some $i \geq 0$, we have $|B(v,i)| \leq n/2$. Then the Cheeger inequality states that

$$e(B(v,i), V(G) \setminus B(v,i)) \geq (d-\lambda)|B(v,i)|/2.$$

In this case, since G is d-regular, get

$$|B(v,i+1)| \geq |B(v,i)| + d^{-1}e(B(v,i), V(G) \setminus B(v,i)) \geq \left(1 + \frac{d-\lambda}{2d}\right)|B(v,r)|.$$

Moreover, if $|B(v,i)| \geq n/2$, then we also have $|B(v,r)| \geq n/2$ for all $r \geq i$. Since $|B(v,0)| = 1$, we obtain by induction that

$$|B(v,r)| \geq \min\left\{\left(1 + \frac{d-\lambda}{2d}\right)^r, n/2\right\}.$$

Let $r_0 := \left\lceil \log(n/2)/\log\left(1 + \frac{d-\lambda}{2d}\right)\right\rceil$. Then for every vertex v, we have $|B(v,r_0)| \geq n/2$. In particular, any two vertices and have distance at most $2r_0 + 1$ (here we use that since $\lambda < d$, G must be connected). Thus, if we choose the constant $0 \leq c \leq (d-\lambda)/(2d)$ so small that $(1+c)^{2r_0} \leq n/2$ holds for all $n \geq 3$ (such a choice is possible), then we have

$$|B(v,r)| \geq \min\{(1+c)^r, n\}$$

for all vertices $v \in V(G)$ and all $r \geq 0$, provided that G has at least three vertices. Indeed, if $|B(v,r)| \leq n/2$, then this follows from the arguments above. It is also clearly true if $|B(v,r)| = n$. Finally, if $n/2 < |B(v,r)| < n$, then $r \leq 2r_0$ and so it is true because $(1+c)^r \leq (1+c)^{2r_0} \leq n/2 < |B(v,r)|$.

If G has only one or two vertices, it is very easy to check that the same statement still holds.